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The Boltzmann Equation for Bose–Einstein Particles: Velocity Concentration and Convergence to Equilibrium

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Long time behavior of solutions of the spatially homogeneous Boltzmann equation for Bose–Einstein particles is studied for hard potentials with certain cutoffs and for the hard sphere model. It is proved that in the cutoff case solutions as time $t \to \infty$ converge to the Bose–Einstein distribution in L^1 topology with the weighted measure $(\varrho + |v|^2)dv$, where $\varrho = 1$ for temperature $T \geqslant T_c$ and $\varrho = 0$ for $T < T_c$. In particular this implies that if $T < T_c$ then the solutions in the velocity regions $\{v \in \mathbf{R}^3 |\ |v| \leqslant \delta(t)\}$ (with $\delta(t) \to 0$) converge to a unique Dirac delta function (velocity concentration). All these convergence are uniform with respect to the cutoff constants. For the hard sphere model, these results hold also for weak or distributional solutions. Our methods are based on entropy inequalities and an observation that the convergence to Bose–Einstein distributions can be reduced to the convergence to Maxwell distributions.

KEY WORDS: Bose–Einstein particles; temperature, entropy; velocity concentration; convergence to equilibrium.

1. INTRODUCTION

The Boltzmann equation for Bose–Einstein particles under consideration (after normalizing a quantum parameter) is given by

$$\begin{aligned}\frac{\partial}{\partial t} f(v,t) &= \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B(v - v_*, \omega)\big(f f_*'(1+f)(1+f_*) \\ &\quad - f f_*(1+f')(1+f_*')\big)\, d\omega\, dv_* \qquad \text{(BBE)}\end{aligned}$$

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which describes time-evolution of dilute, space homogeneous, and one species Bose gases. Physical background and derivations of this model can be found in Chapman and Cowling (9, Chap. 17), Nordheim,[23] and Uehling and Uhlenbeck.[31] In kinetic theory, Eq. (BBE) and its modifications attract research interests in their amazing quantum effects: For very low temperatures, such models can be used to describe the process of the formation of the Bose–Einstein condensation (i.e. the velocity concentration), see e.g. Semikov and Tkachev.[27,28]

Before going to main subject, let us introduce notations. In Eq. (BBE), a solution $f(v,t)$ is the density of the number of particles at time $t\in[0,\infty)$ with the velocity $v\in\mathbf{R}^3$ (therefore $f(v,t)$ should be $\geqslant 0$), and f_*, f', f'_* denote the same function f at velocity variables v_*, v', v'_* respectively, i.e.

$$f=f(v,t),\quad f_*=f(v_*,t),\quad f'=f(v',t),\quad f'_*=f(v'_*,t),$$

where v, v_* and v'_*, v'_* are velocities of two particles before and after their collision; they have the following relations due to the conservations of momentum and kinetic energy:

$$v'+v'_*=v+v_*,\quad |v'|^2+|v'_*|^2=|v|^2+|v_*|^2 \tag{1.1}$$

which can be solved explicitly as

$$v'=v-((v-v_*)\cdot\omega)\omega,\quad v'_*=v_*+((v-v_*)\cdot\omega)\omega,\quad \omega\in\mathbf{S}^2, \tag{1.2}$$

where $u\cdot v$ denotes the scale product in $\mathbf{R}^3$, $|u|=\sqrt{u\cdot u}$, and $\mathbf{S}^2=\{\omega\in\mathbf{R}^3 \,|\, |\omega|=1\}$. (1.2) implies $|(v'-v'_*)\cdot\omega|=|(v-v_*)\cdot\omega|$ and $|v'-v'_*|=|v-v_*|$ which as well as (1.1)–(1.2) are often used in the fundamental operations of the Boltzmann collision integrals.[6,7]

The collision kernel $B(v-v_*,\omega)\equiv B(|v-v_*|,\cos\theta)$ is a non-negative Borel function of $(|v-v_*|,\cos\theta)$ only, where $\cos\theta=|\mathbf{n}\cdot\omega|, \mathbf{n}=(v-v_*)/|v-v_*|, \theta\in[0,\pi/2]$. For molecular interaction potentials with the inverse-power law, B takes the form

$$B(v-v_*,\omega)=b(\cos\theta)|v-v_*|^\gamma.$$

In this paper we mainly consider the hard potentials and the hard sphere model: $0<\gamma\leqslant 1$. For the hard sphere model, B is given by $b(\tau)=\text{const.}\tau$ and $\gamma=1$, i.e.

$$B(v-v_*,\omega)=\text{const.}\cos(\theta)|v-v_*| \tag{1.3}$$

which is the only "non-cutoff" kernel satisfying $\int_0^1 b(\tau)d\tau<\infty$.[6]

Function spaces for solutions in the velocity variable are usually chosen weighted Lebesgue spaces $L^1_s(\mathbf{R}^3)$ $(s\geqslant 0)$ which are defined for measurable functions by

$$f\in L^1_s(\mathbf{R}^3)\Longleftrightarrow \|f\|_{L^1_s}:=\int_{\mathbf{R}^3}(1+|v|^2)^{s/2}|f(v)|\,dv<\infty$$

and we denote $L^1_0(\mathbf{R}^3)=L^1(\mathbf{R}^3)$ for $s=0$. To study velocity concentration at $v=0$ we also use the $|v|^s$-weighted norm (for $s>0$)

$$\|f\|_{L^1_s(\backslash 0)}:=\int_{\mathbf{R}^3}|v|^s|f(v)|\,dv.$$

The condition on the L^1-initial data $f|_{t=0}=f_0$ is given by $0\leqslant f_0\in L^1_2(\mathbf{R}^3)$ which means that the mass and kinetic energy of the gas system (per unit space volume) are finite and thus the entropy is also finite (see Lemma 2). Since after a velocity translation the function $f(v+v_0,t)$ is still a solution of Eq. (BBE) with the initial datum $f_0(v+v_0)$, we can assume that the mean velocity v_0 is zero. This is equivalent to assuming that $\int_{\mathbf{R}^3} vf_0(v)dv=0$.

Classical calculation using (1.1)–(1.2) shows at least formally that a solution $f(v,t)$ of Eq. (BBE) satisfies the conservation of the mass, momentum, and energy (such a solution will be called a conservative solution):

$$\int_{\mathbf{R}^3}(1,v,|v|^2)f(v,t)dv=\int_{\mathbf{R}^3}(1,v,|v|^2)f_0(v)dv$$

and the entropy identity:

$$\frac{d}{dt}S(f(t))=\frac{1}{4}D(f(t)),$$

where $f(t)\equiv f(\cdot,t)$, $S(f)$ is the entropy functional:

$$S(f)=\int_{\mathbf{R}^3}\Big((1+f)\log(1+f)-f\log f\Big)dv \tag{1.4}$$

$D(f)$ is the entropy dissipation:

$$D(f)=\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2}B(v-v_*\omega)\Gamma(f)d\omega\,dv_*\,dv$$

and

$$\Gamma(f) = \Gamma\left(f' f'_*(1+f)(1+f_*),\, f f_*(1+f')(1+f'_*)\right). \tag{1.5}$$

Here

$$\Gamma(x,y) = (x-y)\log\left(\frac{x}{y}\right) m \quad x, y \geqslant 0 \tag{1.6}$$

and we define $\Gamma(x,y)=\infty$ for $x>0=y$ or $0=x<y$; $\Gamma(x,y)=0$ for $x=y=0$.

Mathematical results on the BBE model are so far rather incomplete due to its strong non-linearity. In ref. 20, under a strong cutoff assumption on the kernel B (see (6.1)) and assuming the initial data are isotropic, $f_0(v)=f_0(|v|)$, we proved the global existence and uniqueness of conservative solutions of Eq. (BBE) and obtained some weak results on the convergence to equilibrium sates and the velocity concentration. With a similar cutoff assumption on B, the results on global existence and uniqueness of conservative isotropic solutions were extended to distributional (i.e. measure-valued) isotropic solutions by Escobedo *et al.*[14] using rigorously Dirac delta functions. By establishing a weak form of Eq. (BBE) and a weak stability of distributional solutions we extended the existence result of ref. 14 to including the hard sphere model.[22] For similar quantum kinetic models we refer to Caffish and Levermore,[3] Escobedo *et al.*[11] (for Kompaneets equation), Escobedo and Mischler,[12] Escobedo *et al.*[15] (for Boltzmann–Compton equation), and Escobedo[13] (for both equations). It is also profit to see Saint-Raymond[26] and Villani (32, Chap. 5) for reviews and questions about such quantum kinetic models.

The present paper is a continuation of refs. 20, 22: We will prove the strong convergence to equilibrium and the single point concentration (as $t\to\infty$) for L^1 solutions $f(v,t)$ of Eq. (BBE) with the cutoff assumption (6.1) and for distributional solutions F_t for the hard sphere model. As mentioned above, the existence of distributional solutions of Eq. (BBE) with the hard sphere model has been proven in ref. 22 under the isotropic condition which means that the initial data and therefore the solutions can be represented (in terms of Riesz representation theorem) by finite positive Borel measures on $\mathbf{R}_+ := [0,\infty)$. In this paper the isotropic condition for distributional solutions is only used to insure the existence of conservative solutions which are obtained by approximated L^1 isotropic solutions having the moment production property (6.5). Except this, we just need the initial data F_0 and hence the solutions F_t belong to $\mathcal{B}_2^+(\mathbf{R}^3)$. Here $\mathcal{B}_2^+(\mathbf{R}^3)$

is the class of finite positive Borel measures F on $\mathbf{R}^3$ satisfying

$$\int_{\mathbf{R}^3}(1+|v|^2)dF(v)<\infty.$$

Our main idea comes from an observation that the strong convergence of solution $f(v,t)$ to the Bose–Einstein distribution $\Omega_{a,b}(v) = ae^{-b|v|^2}/(1-ae^{-b|v|^2})$ (an equilibrium solution of Eq. (BBE)) can be reduced to the strong convergence of $f(v,t)/(1+f(v,t))$ to the Maxwell distribution $M_{a,b}(v)=ae^{-b|v|^2}$. Techniques such as entropy dissipation, moment production, and the strong compactness of the classical Boltzmann gain term $Q^+(\cdot,\cdot)$ can all work with that reduction. Also we obtain Csiszár–Kullback–Pinsker type inequalities and an inverse inequality for the BBE model, which imply that for any temperature ($T\geqslant T_c$ or $T<T_c$)

$$\|f(t)-\Omega_{a,b}\|_{L^1_2(\setminus 0)}\to 0 \Longleftrightarrow S(f(t))\to S(\Omega_{a,b}). \tag{1.7}$$

And for $T\geqslant T_c$,

$$\|f(t)-\Omega_{a,b}\|_{L^1_2}\to 0 \Longleftrightarrow S(f(t))\to S(\Omega_{a,b}). \tag{1.8}$$

We now outline the key parts which motivate our proofs for the strong convergence and the velocity concentration. In the following the constants C are independent of t because of the conservation laws. Let $g=\frac{f}{1+f}$, $f=f(v,t)$. Applying the inequality

$$|x-y|\leqslant\sqrt{x+y}\sqrt{\Gamma(x,y)},\quad x,y\geqslant 0 \tag{1.9}$$

to $x=g'g'_*$, $y=gg_*$ and using Cauchy–Schwarz inequality we find the following control:

$$\int_{\mathbf{R}^3}(1+f)\iint_{\mathbf{R}^3\times\mathbf{S}^2}B_{\min}|g'g'_*-gg_*|\,d\omega\,dv_*\,dv\leqslant C\sqrt{D_{\min}(f)}, \tag{1.10}$$

where $B_{\min}$ is a suitable "minimum" kernel constructed from B and $D_{\min}(f)$ is the entropy dissipation corresponding to $B_{\min}$; they satisfy $B_{\min}\leqslant B$ hence $D_{\min}(f)\leqslant D(f)$, and the relevant collision integrals with the kernel $B_{\min}$ are all bounded. The role of the factor $1+f$ in (1.10) is to produce a control for $|f-\Omega|$ in some weighted L^1 topology. Here $\Omega(v)=M(v)/(1-M(v))$ is a general Bose–Einstein distribution with $M(v)=$

$ae^{-b|v-v_0|^2}$ and $0<a\leqslant 1, 0<b<\infty$. We observe that f, Ω and g, M have a simple but important relation:

$$(1-M)(f-\Omega)=(1+f)(g-M). \tag{1.11}$$

To see the connection between (1.10) and (1.11) we need the classical Boltzmann collision operators $Q^{\pm}(\varphi,\psi)$ which are given for a general kernel B by

$$Q^{+}(\varphi,\psi)(v)=\int\int_{\mathbf{R}^3\times\mathbf{S}^2} B(v-v_*,\omega)\varphi(v')\psi(v'_*)d\omega\, dv_*, \tag{1.12}$$

$$Q^{-}(\varphi,\psi)(v)=\varphi(v)L(\psi)(v), \quad L(\psi)(v)=\int_{\mathbf{R}^3} A(v-v_*)\psi(v_*)dv_*, \tag{1.13}$$

$$A(v-v_*)=A(|v-v_*|)=\int_{\mathbf{S}^2} B(v-v_*,\omega)d\omega. \tag{1.14}$$

Now let B be replaced by $B_{\min}$. The strict positivity of $M(v)$ implies that $L(M)(v)$ has a positive lower bound: $L(M)(v)\geqslant c>0$ from which and the identity (1.11) we deduce

$$\begin{aligned}\|(1-M)(f-\Omega)\|_{L^1} &\leqslant \frac{1}{c}\|(1+f)[gL(M)-ML(M)]\|_{L^1}\\ &\leqslant \frac{1}{c}\|(1+f)[Q^-(g,g)-Q^+(g,g)]\|_{L^1}+\text{easy term.}\end{aligned}$$

The first term in the right-hand side is bounded by $\sqrt{D_{\min}(f)}$ due to (1.10), while the "easy term" is bounded by $C\|g-M\|_{L^1}$ because of the identity $Q^-(M,M)=Q^+(M,M)$ for Maxwellian M. These yield a nice control:

$$\|(1-M)(f-\Omega)\|_{L^1}\leqslant C\left(\|g-M\|_{L^1}+\sqrt{D_{\min}(f)}\right). \tag{1.15}$$

On the other hand, applying the moment production property and following Lions' argument[18] using the compactness of $\{Q^+(g,g)(\cdot,t)\}_{t\geqslant 1}$ and the entropy dissipation decay $D_{\min}(f(t_n))\to 0$ $(t_n\to\infty)$ we prove that there exists a Maxwellian M such that $\|g(t_n)-M\|_{L^1}\to 0$ as $t_n\to\infty$. And any such M, or equivalently $\Omega=M/(1-M)$, are in fact unique, determined by the initial state f_0 with mean velocity $v_0=0$, i.e. $M=M_{a,b}$, $\Omega=\Omega_{a,b}$. By (1.15), this implies

$$\|(1-M_{a,b})(f(t)-\Omega_{a,b})\|_{L^1}\to 0 \quad (t\to\infty).$$

This convergence holds also in L^1_2 norm (thanks to the moment production property) and thus by inequality $|v|^2 \leqslant C(1+|v|^2)(1-M_{a,b}(v))$ we obtain

$$\|f(t)-\Omega_{a,b}\|_{L^1_2(\setminus 0)} \to 0 \quad (t\to\infty). \tag{1.16}$$

By (1.7) this gives the entropy convergence $S(f(t)) \to S(\Omega_{a,b})$ which in turn implies the L^1_2 convergence in (1.8) for $T \geqslant T_c$.

On the other hand, at low temperature $T<T_c$, which implies $a=1$, we have

$$\frac{1}{N}\int_{\mathbf{R}^3} \Omega_{1,b}(v)dv = (T/T_c)^{3/2} < 1,$$

where $N=\int_{\mathbf{R}^3} f_0(v)dv$. Applying (1.16) we then obtain the Bose–Einstein condensation in the sense that, as $t\to\infty$,

$$\frac{1}{N}\int_{|v|\leqslant \delta(t)} f(v,t)dv \to 1-(T/T_c)^{3/2}, \tag{1.17}$$

where $\delta(t)=(\|f(t)-\Omega_{1,b}\|_{L^1_2(\setminus 0)})^{1/4} \to 0 (t\to\infty)$.

All these results hold also for distributional solutions by taking weak limit for approximate L^1 solutions $f^n(v,t)$ and by using the monotonicity of the entropy $t\mapsto S(f^n(t))$.

In ref. 20 the limit of velocity concentration was only proved for $\int_{v\in S_t} f(v,t)dv$ where S_t are abstract sets satisfying $mes(S_t) \to 0 (t\to\infty)$. This is because we used a very abstract argument: the biting-weak convergence.[1] The present work catches the feature of the convergence problem and reduces the single point concentration (1.17) to the strong convergence to Bose–Einstein distribution $\Omega_{1,b}$ in the norm $\|\cdot\|_{L^1_2(\setminus 0)}$, but the method is not yet completely constructive because our compactness argument could not give any explicit convergence rate. In order to obtain an explicit convergence rate for the BBE model one may study and generalize the modern entropy dissipation techniques which deal with the rate of convergence to Maxwellians for solutions of the classical Boltzmann equation (see for instance Carlen and Carvalho,[4,5] Cercignani,[8] Desvillettes,[10] Toscani and Villani,[29] Villaini[33]). This is of course very difficult but seems possible in terms of the Csiszár–Kullback–Pinsker type inequalities (4.15)–(4.16), the inequality (1.15) or (3.6), and the L^∞-bound $0\leqslant g=f/(1+f)\leqslant 1$. Finally we note that our proof of the single point concentration (1.17) is given only in the sense $t\to\infty$. This is because that for L^1 solutions with the strong cutoff assumption (6.1), any kind of velocity concentration (single or non-single point) can not happen within finite time.[20] For distributional solutions for the hard sphere model, it is not known whether the single point concentration can occur in finite time. To obtain such a concentration at a time $t_c < \infty$, one may consider (as suggested by the above estimates) to prove that $\|f_t - \Omega_{1,b}\|_{L^1_2(\setminus 0)} \to 0$ or equivalently $S(f_t) \to S(\Omega_{1,b})$ as $t \to t_c$, where f_t is the L^1 part of F_t (see Theorem 2). But this seems rather difficult though may not be completely impossible. For the Kompaneets equation, it has been proven in ref. 11 that concentrations can appear in finite time. See also ref. 26 for comments about this kind of researches.

The rest of the paper is organized as follows: In Section 2 we discuss temperatures and the Bose–Einstein distributions and give some relations between them. In Section 3 we realize the reduction from the "Bose–Einstein convergence" to the "Maxwellian convergence". In Section 4 we prove some inequalities about entropy. Section 5 is a further preparation for Sections 6 and 7. In Section 6 we prove the strong convergence to Bose–Einstein distributions and the velocity concentration for L^1 solutions as shown above. Finally in Section 7 we apply the results of Sections 5 and 6 to prove the strong convergence and the concentration for distributional solutions for the hard sphere model.

2. ABOUT TEMPERATURE AND EQUILIBRIUM

For a spatially homogeneous Bose gas system governed by Eq. (BBE) with mean velocity zero, let $0 < N, E < \infty$ be such that mN and $\frac{1}{2}mE$ are the total mass and kinetic energy per unit space volume, where m is the mass of a particle. According to (9, Chap. 2) and (30, pp. 43–44), the kinetic temperature of the gas system is defined by

$$\overline{T} = \frac{m}{2} \cdot \frac{E}{N} \cdot \frac{2}{3k_{\rm B}},$$

where $k_{\rm B} > 0$ is the Boltzmann constant. In order to study the Bose–Einstein condensation, we will use the critical kinetic temperature

$$\overline{T}_c = \frac{\zeta(5/2)}{\zeta(3/2)} T_c,$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s} (s > 1)$ is the Riemann–Zeta function,

$$T_c = \frac{m}{2\pi k_B}\left(\frac{N}{\zeta(3/2)}\right)^{2/3} = \frac{h^2}{2\pi m k_B}\left(\frac{\mathcal{N}}{\zeta(3/2)\mathrm{g}}\right)^{2/3}, \tag{2.1}$$

$\mathcal{N} = (m/h)^3 \mathrm{g} N$, h is the Planck constant, and g is a statistical weight. Recall that the second equality in (2.1) is often used in text books for defining T_c. See for instance Landau and Lifshitz (17, pp. 180–181, p. 36), Parthia (24, p.180 for $\mathrm{g} = 1$ (spinless)).

A Bose–Einstein distribution $\Omega_{a,b}$ in $L^1(\mathbf{R}^3)$ (with mean velocity zero) is an equilibrium solution of Eq. (BBE). Because of the integrability, $\Omega_{a,b}$ has the unique form

$$\Omega_{a,b}(v) = \frac{ae^{-b|v|^2}}{1 - ae^{-b|v|^2}}, \quad 0 < a \leqslant 1, \quad 0 < b < \infty \tag{2.2}$$

For a Bose gas system with the numbers N, E, it is physically required that the corresponding Bose–Einstein distribution $\Omega_{a,b}$ is the maximizer of the following conditional maximum problem:

$$\int_{\mathbf{R}^3} \Omega_{a,b}(v)dv = \max_{\tilde{a},\tilde{b}} \int_{\mathbf{R}^3} \Omega_{\tilde{a},\tilde{b}}(v)dv, \tag{2.3}$$

where $\tilde{a}, \tilde{b}$ under the max satisfy

$$\int_{\mathbf{R}^3} \Omega_{\tilde{a},\tilde{b}}(v)dv \leqslant N, \quad \int_{\mathbf{R}^3} \Omega_{\tilde{a},\tilde{b}}(v)|v|^2 dv = E, \quad 0 < \tilde{a} \leqslant 1, \quad 0 < \tilde{b} < \infty. \tag{2.4}$$

The coefficients of $\Omega_{a,b}$ in (2.2) can be solved uniquely through N and E. To see this we introduce the functions

$$\mathrm{R}(t) = \frac{\mathrm{P}_{5/2}(t)}{[\mathrm{P}_{3/2}(t)]^{5/3}}, \quad \mathrm{P}_s(t) = \sum_{n=1}^{\infty} \frac{t^n}{n^s}, \qquad t \in (0, 1], \ \ s > 1.$$

It has been proven in ref. 20 that the function $\mathrm{R}(t)$ is strictly decreasing on $t \in (0, 1]$ and $\mathrm{R}(t) \to \infty$ as $t \to 0+$. For **any** $\tilde{a}, \tilde{b}$ satisfying (2.4) we compute (see ref. 20)

$$N_e(\tilde{a}, \tilde{b}) := \int_{\mathbf{R}^3} \Omega_{a,b}(v)dv = \left(\frac{1}{\tilde{b}}\right)^{3/2} \pi^{3/2} \mathrm{P}_{3/2}(\tilde{a}), \tag{2.5}$$

$$E = \int_{\mathbf{R}^3} \Omega_{\tilde{a},\tilde{b}}(v)|v|^2 dv = \left(\frac{1}{\tilde{b}}\right)^{5/2} \frac{3}{2} \pi^{3/2} \mathrm{P}_{5/2}(\tilde{a}). \tag{2.6}$$

These imply

$$\frac{E}{[N_e(\tilde{a}, \tilde{b})]^{5/3}} = \frac{3}{2\pi} \mathrm{R}(\tilde{a}).$$

By definition of $\overline{T}, \overline{T}_c$ and notice that $\mathrm{R}(1) = \zeta(5/2)/[\zeta(3/2)]^{5/3}$ we deduce

$$\frac{1}{N}\int_{\mathbf{R}^3} \Omega_{\tilde{a},\tilde{b}}(v)dv = \frac{N_e(\tilde{a},\tilde{b})}{N} = \left(\frac{\overline{T}}{\overline{T}_c}\cdot\frac{\mathrm{R}(1)}{\mathrm{R}(\tilde{a})}\right)^{3/5}. \tag{2.7}$$

Now let a, b be such that $\Omega_{a,b}$ is the Bose–Einstein distribution satisfying (2.2)–(2.4). Then applying the relation (2.7) and noting that $\mathrm{R}(t) \geqslant \mathrm{R}(1)$ for $0 < t \leqslant 1$, we obtain the following identities for all $\overline{T}/\overline{T}_c$:

$$\frac{1}{N}\int_{\mathbf{R}^3} \Omega_{a,b}(v)dv = \left(\frac{\overline{T}}{\overline{T}_c}\cdot\frac{\mathrm{R}(1)}{\mathrm{R}(a)}\right)^{3/5} = \min\left\{\left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5}, 1\right\}. \tag{2.8}$$

In order to connect equilibrium statistical physics, we also represent $\Omega_{a,b}$ in terms of the thermodynamic temperature $T(>0)$ (refs. 9, 16, 17) by

$$\Omega_{a,b}(v) = \frac{1}{e^{\alpha+\beta m|v|^2/2} - 1}, \quad \beta = \frac{1}{k_{\mathrm{B}}T}$$

with the change

$$a = e^{-\alpha}, \quad \alpha \geqslant 0, \quad b = \beta m/2.$$

The two temperatures $\overline{T}$ and T have the following relation (see ref. 22):

$$\left(\frac{T}{T_c}\right)^{3/2} = \left(\frac{\overline{T}}{\overline{T}_c}\right)^{3/5}\left(\frac{\mathrm{P}_{5/2}(1)}{\mathrm{P}_{5/2}(a)}\right)^{3/5}. \tag{2.9}$$

Let $\mathrm{R}^{-1}(\tau)$ with $\tau \in [\mathrm{R}(1), \infty)$ be the inverse function of $\mathrm{R}(t)$. By simple calculation using (2.7), (2.8), and (2.9) one can easily prove the following

Proposition 1. Let $\Omega_{a,b}$ be the Bose–Einstein distribution satisfying (2.2)–(2.4). Then (1)

$$\overline{T} \geqslant \overline{T}_c \Longleftrightarrow \int_{\mathbf{R}^3} \Omega_{a,b}(v)dv = N \quad \text{and} \quad \int_{\mathbf{R}^3} |v|^2\Omega_{a,b}(v)dv = E.$$

In that case,

$$a = \mathrm{R}^{-1}\left(\mathrm{R}(1)\overline{T}/\overline{T}_c\right), \quad b = \left(\frac{1}{E}\cdot\frac{3}{2}\pi^{3/2}\mathrm{P}_{5/2}(a)\right)^{2/5}.$$

(2)

$$\overline{T} < \overline{T}_c \Longleftrightarrow a = 1, \quad \int_{\mathbf{R}^3} \Omega_{1,b}\, dv < N \quad \text{and} \quad \int_{\mathbf{R}^3} |v|^2 \Omega_{1,b}\, dv = E.$$

In that case,

$$b = \left(\frac{1}{E} \cdot \frac{3}{2} \pi^{3/2} \mathrm{P}_{5/2}(1) \right)^{2/5} = \left(\frac{1}{E} \cdot \frac{3}{2} \pi^{3/2} \zeta(5/2) \right)^{2/5}.$$

(3)

$$\overline{T} \geqslant \overline{T}_c \Longleftrightarrow T \geqslant T_c.$$

$$\overline{T} \leqslant \overline{T}_c \quad \text{or} \quad T \leqslant T_c \Longleftrightarrow (T/T_c)^{3/2} = (\overline{T}/\overline{T}_c)^{3/5}.$$

As one sees that at low kinetic temperature $\overline{T} < \overline{T}_c$, there is a mass gap:

$$N_0 := N - \int_{\mathbf{R}^3} \Omega_{1,b}(v) dv = N(1 - (\overline{T}/\overline{T}_c)^{3/5}) > 0. \tag{2.10}$$

In view of conservation laws, the Bose–Einstein distribution as a final and equilibrium state of a Bose gas system should have the same mass as that of the initial state f_0 for any temperature. In this sense, the Bose–Einstein distribution for low kinetic temperature $\overline{T} < \overline{T}_c$ should be defined as $\Omega_{1,b}(v) + N_0 \delta_0(v)$ which is indeed an equilibrium solution of Eq. (BBE) in a weak form (see ref. 22), where $\delta_0(v)$ is the standard Dirac delta function. In this paper, our study on long time behavior of solutions of Eq. (BBE) is focused on the L^1 distribution $\Omega_{a,b}$ because the the final velocity concentration in the ratio $N_0/N = 1 - (\overline{T}/\overline{T}_c)^{3/5}$ is proved only through the strong convergence of solutions to $\Omega_{1,b}$. Thus for convenience we still call $\Omega_{1,b}$ a Bose–Einstein distribution.

3. REDUCTION TO MAXWELLIAN CONVERGENCE

In this section we will use the classical collision operators $Q^{\pm}(\varphi, \psi)$ given in (1.12)–(1.14). The main result of this section is Proposition 2. We begin by the following

Lemma 1. Let a collision kernel B satisfy

$$0 \leqslant B(v - v_*, \omega) \leqslant \underline{b}(\cos\theta)\cos^3\theta\sin^3\theta, \tag{3.1}$$
$$A_0 := 4\pi \int_0^{\pi/2} \sin\theta \underline{b}(\cos\theta) d\theta < \infty. \tag{3.2}$$

Then for any $\varphi, \psi \in L^1(\mathbf{R}^3)$ satisfying $0 \leqslant \varphi, \psi \leqslant 1$ we have

$$Q^+(\varphi, \psi)(v) \leqslant A_0 \min\{\|\varphi\|_{L^1}, \|\psi\|_{L^1}\}, \quad v \in \mathbf{R}^3. \tag{3.3}$$

Proof. By Lemma 1 in ref. 21 we have

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} \underline{b}(\cos\theta)\cos^3\theta\, \sin^3\theta\varphi(v')d\omega\, dv_*$$
$$= \left(4\pi \int_0^{\pi/2} \sin\theta \underline{b}(\cos\theta)\sin^3\theta\, d\theta\right) \|\varphi\|_{L^1} \leqslant A_0\|\varphi\|_{L^1},$$

$$\iint_{\mathbf{R}^3\times\mathbf{S}^2} \underline{b}(\cos\theta)\cos^3\theta\sin^3\theta\psi(v'_*)d\omega dv_*$$
$$= \left(4\pi \int_0^{\pi/2} \sin\theta \underline{b}(\cos\theta)\cos^3\theta\, d\theta\right) \|\psi\|_{L^1} \leqslant A_0\|\psi\|_{L^1}.$$

By assumption (3.1) and the inequality $0 \leqslant \varphi(v')\psi(v'_*) \leqslant \min\{\varphi(v'), \psi(v'_*)\}$, these imply (3.3). ∎

Proposition 2. Let $B(v - v_*, \omega) = \underline{b}(\cos\theta)\cos^3\theta\sin^3\theta\Psi(|v - v_*|)$ where $\underline{b}(\cdot) \geqslant 0$ satisfies (3.2) and

$$A_0^* := 4\pi \int_0^{\pi/2} \sin\theta \underline{b}(\cos\theta)\cos^3\theta\, \sin^3\theta\, d\theta > 0 \tag{3.4}$$

$\Psi(r)$ is non-decreasing on $r \in [0, \infty)$ with $0 < \Psi(r) \leqslant 1$ for all $r > 0$. Let $0 < a \leqslant 1, 0 < b < \infty, v_0 \in \mathbf{R}^3$ be constants and let

$$M(v) = ae^{-b|v-v_0|^2}, \quad \Omega(v) = \frac{M(v)}{1 - M(v)}.$$

Then for any $0 \leqslant f \in L^1(\mathbf{R}^3)$, $g = \frac{f}{1+f}$ we have

$$\int_{\mathbf{R}^3} (1+f) \iint_{\mathbf{R}^3 \times \mathbf{S}^2} B|g'g'_* - gg_*| d\omega \, dv_* dv \leqslant \sqrt{3A_0}\, \|f\|_{L^1} \sqrt{D(f)} \quad (3.5)$$

and

$$\|(1-M)(f-\Omega)\|_{L^1} \leqslant C_M (\|f\|_{L^1} + \|M\|_{L^1}) \left(\|g - M\|_{L^1} + \sqrt{D(f)} \right), \quad (3.6)$$

where $D(f)$ is the entropy dissipation corresponding to $B(v - v_*, \omega)$,

$$C_M = C_{A_0, A_0^*, \Psi, M} = \frac{8(A_0 + \sqrt{A_0})}{A_0^*} \left(\int_{\mathbf{R}^3} a e^{-b|z|^2} \Psi(|z|) dz \right)^{-1}.$$

Proof. By definition of $A(z)$ given in (1.14) we have $A(z) = A(|z|) = A_0^* \Psi(|z|)$. By assumption on Ψ, $A(r) = A_0^* \Psi(r)$ is non-decreasing on $r \in [0, \infty)$ from which and the spherical transformation we deduce a positive lower bound of $L(M)$: For all $v \in \mathbf{R}^3$,

$$\begin{aligned} L(M)(v) &= \int_{\mathbf{R}^3} M(v_*) A(v - v_*) dv_* = \int_{\mathbf{R}^3} a e^{-b|v_* - v_0|^2} A(|v - v_*|) dv_* \\ &= \int_{\mathbf{R}^3} a e^{-b|z|^2} A(|v - v_0 - z|) dz \geqslant \frac{1}{2} \int_{\mathbf{R}^3} a e^{-b|z|^2} A(\sqrt{|v - v_0|^2 + |z|^2}) dz \\ &\geqslant \frac{1}{2} \int_{\mathbf{R}^3} a e^{-b|z|^2} A(|z|) dz = \frac{1}{2} A_0^* \int_{\mathbf{R}^3} a e^{-b|z|^2} \Psi(|z|) dz := \frac{1}{2} C_{A,M} > 0. \end{aligned}$$

Here the strict positivity $C_{A,M} > 0$ is because $A_0^* > 0, a > 0$ and $\Psi(|z|) > 0$ for $z \neq 0$.

From this lower bounds and the identity (1.11) and using the fact that $ML(M) = Q^-(M, M) = Q^+(M, M)$ and $(1+f)g = f$, we have

$$\begin{aligned} &\frac{1}{2} C_{A,M} (1-M)|f - \Omega| = \frac{1}{2} C_{A,M} (1+f)|g - M| \\ &\quad \leqslant L(M)(1+f)|g - M| = (1+f)|gL(M) - ML(M)| \\ &\quad \leqslant (1+f)g|L(M) - L(g)| + (1+f)|gL(g) - ML(M)| \\ &\quad \leqslant f|L(M-g)| + (1+f)|Q^-(g,g) - Q^+(g,g)| \\ &\qquad + (1+f)|Q^+(g,g) - Q^+(M,M)|. \end{aligned}$$

Thus

$$
\begin{aligned}
\frac{1}{2} C_{A,M}\|(1-M)(f-\Omega)\|_{L^1} \leqslant \|f[L(M-g)]\|_{L^1} \\
+\|(1+f)[Q^-(g,g)-Q^+(g,g)]\|_{L^1} \\
+\|(1+f)[Q^+(g,g)-Q^+(M,M)]\|_{L^1} := I_1+I_2+I_3. \qquad (3.7)
\end{aligned}
$$

By L^∞ bounds $A(v-v_*)\leqslant A_0^*\leqslant A_0$ we have $|L(M-g)(v)|\leqslant A_0\|g-M\|_{L^1}$ for all $v\in\mathbf{R}^3$. This implies

$$
I_1 = \|f[L(M-g)]\|_{L^1} \leqslant A_0\|f\|_{L^1}\|g-M\|_{L^1}. \qquad (3.8)
$$

Also we have

$$
\begin{aligned}
I_2 &= \|(1+f)[Q^-(g,g)-Q^+(g,g)]\|_{L^1} \\
&\leqslant \int_{\mathbf{R}^3}(1+f)\iint_{\mathbf{R}^3\times\mathbf{S}^2} B|g'g'_*-gg_*|d\omega\, dv_*\, dv. \qquad (3.9)
\end{aligned}
$$

Let $\Gamma(f)$ and $\Gamma(x,y)$ be the functions given in (1.5) and (1.6). Let

$$
\Pi(f)(v,v_*,\omega)=(1+f)(1+f_*)(1+f')(1+f'_*).
$$

Applying inequality (1.9) we have

$$
|g'g'_*-gg_*|\leqslant\sqrt{g'g'_*+gg_*}\sqrt{\Gamma(g'g'_*,gg_*)}.
$$

Since

$$
(1+f)\Gamma(g'g'_*,gg_*)\leqslant\Pi(f)\Gamma(g'g'_*,gg_*)=\Gamma(f)
$$

this gives

$$
(1+f)|g'g'_*-gg_*|\leqslant\sqrt{(1+f)(g'g'_*+gg_*)}\sqrt{\Gamma(f)}.
$$

Therefore by Cauchy–Schwarz inequality we obtain

$$
\begin{aligned}
&\int_{\mathbf{R}^3}(1+f)\iint_{\mathbf{R}^3\times\mathbf{S}^2} B|g'g'_*-gg_*|d\omega\, dv_*\, dv \\
&\leqslant\left(\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B(1+f)(g'g'_*+gg_*)d\omega\, dv\, dv_*\right)^{1/2} \\
&\quad\times\left(\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\Gamma(f)d\omega\, dv\, dv_*\right)^{1/2} \\
&=\left(\|(1+f)[Q^+(g,g)+Q^-(g,g)]\|_{L^1}\right)^{1/2}\sqrt{D(f)}. \qquad (3.10)
\end{aligned}
$$

Next we compute (using $(1+f)g=f$ and $g\leqslant f$)

$$\begin{aligned}&\|(1+f)[Q^+(g,g)+Q^-(g,g)]\|_{L^1}\\&\quad=\|Q^+(g,g)\|_{L^1}+\|fQ^+(g,g)\|_{L^1}+\|(1+f)gL(g)\|_{L^1}\\&\quad\leqslant 2\|fL(f)\|_{L^1}+\|fQ^+(g,g)\|_{L^1}.\end{aligned}$$

Since $A(|v-v_*|)\leqslant A_0 \Longrightarrow \|fL(f)\|_{L^1}\leqslant A_0(\|f\|_{L^1})^2$ and Lemma 1 applied to $Q^+(g,g)$ leads to

$$\|fQ^+(g,g)\|_{L^1}\leqslant A_0\|f\|_{L^1}\|g\|_{L^1}\leqslant A_0\left(\|f\|_{L^1}\right)^2,$$

it follows that

$$\left(\|(1+f)[Q^+(g,g)+Q^-(g,g)]\|_{L^1}\right)^{1/2}\leqslant\sqrt{3A_0}\|f\|_{L^1}$$

which together with (3.10) proves (3.5). Also by (3.9) and (3.5) we have

$$I_2\leqslant\sqrt{3A_0}\|f\|_{L^1}\sqrt{D(f)}. \tag{3.11}$$

To estimate I_3, we first look at the integrand. Applying Lemma 1 to the functions $0\leqslant g, M, |g-M|\leqslant 1$ we obtain for all $v\in\mathbf{R}^3$

$$\begin{aligned}&|Q^+(g,g)(v)-Q^+(M,M)(v)|\\&\quad\leqslant Q^+(|g-M|,g)(v)+Q^+(M,|g-M|)(v)\leqslant 2A_0\|g-M\|_{L^1}.\end{aligned}$$

Also we have

$$\begin{aligned}&\|Q^+(g,g)-Q^+(M,M)\|_{L^1}\\&\quad\leqslant\|[Q^+(|g-M|,g)\|_{L^1}+\|Q^+(M,|g-M|)\|_{L^1}\\&\quad=\|Q^-(|g-M|,g)\|_{L^1}+\|Q^-(M,|g-M|)\|_{L^1}\\&\quad=\|(g-M)L(g)\|_{L^1}+\|ML(|g-M|)\|_{L^1}\\&\quad\leqslant A_0(\|g\|_{L^1}+\|M\|_{L^1})\|g-M\|_{L^1}.\end{aligned}$$

Therefore

$$\begin{aligned}I_3&=\|(1+f)[Q^+(g,g)-Q^+(M,M)]\|_{L^1}\\&=\|Q^+(g,g)-Q^+(M,M)\|_{L^1}+\|f[Q^+(g,g)-Q^+(M,M)]\|_{L^1}\\&\leqslant A_0(\|g\|_{L^1}+\|M\|_{L^1})\|g-M\|_{L^1}+\|f\|_{L^1}2A_0\|g-M\|_{L^1}\\&\leqslant A_0\left(3\|f\|_{L^1}+\|M\|_{L^1}\right)\|g-M\|_{L^1}.\end{aligned}\tag{3.12}$$

Combining (3.8), (3.11) and (3.12) leads to

$$\begin{aligned}
&\frac{1}{2}C_{A,M}\|(1-M)(f-\Omega)\|_{L^1}\leqslant I_1+I_2+I_3\\
&\leqslant A_0\left(4\|f\|_{L^1}+\|M\|_{L^1}\right)\|g-M\|_{L^1}+\sqrt{3A_0}\|f\|_{L^1}\sqrt{D(f)}\\
&\leqslant 4(A_0+\sqrt{A_0})(\|f\|_{L^1}+\|M\|_{L^1})\left(\|g-M\|_{L^1}+\sqrt{D(f)}\right).
\end{aligned}$$

This proves (3.6) by definition of $C_{A,M}$. ∎

4. SOME INEQUALITIES ABOUT ENTROPY

Recalling (1.4) the entropy $S(f)$ for $0\leqslant f\in L^1_2(\mathbf{R}^3)$ can be written

$$S(f)=\int_{\mathbf{R}^3}s(f(v))dv,$$

where

$$s(y)=(1+y)\log(1+y)-y\log y,\quad y\in[0,\infty) \tag{4.1}$$

which is increasing and concave on $[0,\infty)$:

$$s'(y)=\log(1+\frac{1}{y})>0,\quad s''(y)=-\frac{1}{y(1+y)}<0,\qquad y>0. \tag{4.2}$$

Lemma 2. (1) Let $f,g\in L^1_2(\mathbf{R}^3)$. Then

$$0\leqslant f\leqslant g\Longrightarrow 0\leqslant s(f)\leqslant s(g)\Longrightarrow 0\leqslant S(f)\leqslant S(g). \tag{4.3}$$

(2) If $0\leqslant f\in L^1_2(\mathbf{R}^3)$, then

$$\|s(f)\|_{L^2}\leqslant 2\sqrt{\|f\|_{L^1}},\quad \|s(f)\|_{L^1_1}\leqslant 4\|f\|_{L^1_2}+C_0, \tag{4.4}$$

where C_0 is an absolute constant.

Proof. (4.3) is obvious. To prove (4.4) we write

$$s(f(v)) = \log(1+f(v)) + f(v)\log(1+1/f(v)).$$

Using the inequality $\log(1+y) \leqslant \sqrt{y}$ for $y \geqslant 0$, we get

$$s(f(v)) \leqslant 2\sqrt{f(v)} \quad \forall v \in \mathbf{R}^3.$$

This gives the first inequality in (4.4). Next for any v, by considering $f(v) \geqslant e^{-|v|}$ and $f(v) \leqslant e^{-|v|}$ we obtain

$$s(f(v)) \leqslant 2f(v)(1+|v|) + 2e^{-|v|/2} \quad \forall v \in \mathbf{R}^3.$$

Multiplying $(1+|v|^2)^{1/2}$ to this inequality and taking integration gives the second inequality in (4.4). ∎

Let $F \in \mathcal{B}_2^+(\mathbf{R}^3)$. By Lebesgue–Radon–Nikodym theorem (see e.g. ref. 25), there exist a unique $0 \leqslant f \in L_2^1(\mathbf{R}^3)$, a unique $\mu \in \mathcal{B}_2^+(\mathbf{R}^3)$, and a Borel set $Z \subset \mathbf{R}^3$ satisfying $mes(Z) = 0$, $\mu(\mathbf{R}^3 \backslash Z) = 0$, such that $dF(v) = f(v)dv + d\mu(v)$. We call μ the singular part of F.

Lemma 3. Let $0 \leqslant f^n \in L_2^1(\mathbf{R}^3)$ satisfy $\sup_{n \geqslant 1} \|f^n\|_{L_2^1} < \infty$ and assume that there is an $F \in \mathcal{B}_2^+(\mathbf{R}^3)$ such that

$$\lim_{n\to\infty} \int_{\mathbf{R}^3} \varphi(v) f^n(v)dv = \int_{\mathbf{R}^3} \varphi(v)dF(v) \quad \forall \varphi \in C_c(\mathbf{R}^3). \tag{4.5}$$

Let

$$dF(v) = f(v)dv + d\mu(v),$$

where $0 \leqslant f \in L_2^1(\mathbf{R}^3)$ and $\mu \in \mathcal{B}_2^+(\mathbf{R}^3)$ is the singular part of F. Then

$$\limsup_{n\to\infty} S(f^n) \leqslant S(f) \tag{4.6}$$

Proof. Introduce function space (for constant $s \geqslant 0$)

$$C \cap L^\infty_{-s}(\mathbf{R}^3) = \{\varphi \in C(\mathbf{R}^3) \mid \|\varphi\|_{L^\infty_{-s}} := \sup_{v\in\mathbf{R}^3} |\varphi(v)|(1+|v|^2)^{-s/2} < \infty\}. \tag{4.7}$$

The L^1_2 boundedness of $\{f^n\}$ implies that the convergence in (4.5) holds for all $\varphi \in C \cap L^\infty_{-1}(\mathbf{R}^3)$. By applying convolution it is easy to construct functions $0 \leqslant f_\delta \in C(\mathbf{R}^3) \cap L^1_2(\mathbf{R}^3)$ such that $\|f_\delta - f\|_{L^1_1} \to 0$ as $\delta \to 0+$. Since μ is the singular part of F, there exists a Borel set $Z \subset \mathbf{R}^3$ such that $mes(Z) = 0$ and $\mu(\mathbf{R}^3 \setminus Z) = 0$. For any $\eta > 0$, choose an open set $\mathcal{O}_\eta \supset Z$ such that $mes(\mathcal{O}_\eta) < \eta$. By considering

$$1_{\mathbf{R}^3\setminus\mathcal{O}_\eta}(v) \leqslant \psi_j(v) := \exp(-j\,\mathrm{dist}(v, \mathbf{R}^3\setminus\mathcal{O}_\eta)) \quad \forall\, v \in \mathbf{R}^3, \ \forall\, j \geqslant 1$$

it is easily proven that for any $0 \leqslant \varphi \in C \cap L^\infty_{-1}(\mathbf{R}^3)$

$$\limsup_{n\to\infty} \int_{\mathbf{R}^3\setminus\mathcal{O}_\eta} \varphi(v) f^n(v)dv \leqslant \int_{\mathbf{R}^3\setminus\mathcal{O}_\eta} \varphi(v) f(v)dv. \tag{4.8}$$

Now we will use the following functions

$$\varphi_\varepsilon(v) = \varepsilon e^{-|v|}, \quad \Psi_{\delta,\varepsilon} = s'(f_\delta + \varphi_\varepsilon) = \log\left(1 + \frac{1}{f_\delta + \varphi_\varepsilon}\right)$$

where $0 < \varepsilon \leqslant 1$. It is easily seen that $0 \leqslant \Psi_{\delta,\varepsilon} \in C \cap L^\infty_{-1}(\mathbf{R}^3)$ and $\sup_{\delta>0} \|\Psi_{\delta,\varepsilon}\|_{L^\infty_{-1}} \leqslant C_\varepsilon = 1 + \log(2/\varepsilon) < \infty$. By concavity $s''(y) < 0$ we have

$$\begin{aligned} s(f^n) &\leqslant s(f_\delta + \varphi_\varepsilon) + s'(f_\delta + \varphi_\varepsilon)(f^n - f_\delta - \varphi_\varepsilon) \\ &\leqslant s(f_\delta + \varphi_\varepsilon) + \Psi_{\delta,\varepsilon}(f^n - f) + \Psi_{\delta,\varepsilon}(f - f_\delta), \\ s(f_\delta + \varphi_\varepsilon) &\leqslant s(f + \varphi_\varepsilon) + \log\left(1 + \frac{1}{f + \varphi_\varepsilon}\right)(f_\delta - f) \\ &\leqslant s(f + \varphi_\varepsilon) + C_\varepsilon(1 + |v|^2)^{1/2}|f_\delta - f|. \end{aligned}$$

These imply $s(f^n) \leqslant s(f + \varphi_\varepsilon) + \Psi_{\delta,\varepsilon}(f^n - f) + 2C_\varepsilon(1 + |v|^2)^{1/2}|f_\delta - f|$ and hence

$$\begin{aligned} S(f^n) &= \int_{\mathcal{O}_\eta} s(f^n)dv + \int_{\mathbf{R}^3\setminus\mathcal{O}_\eta} s(f^n)dv \\ &\leqslant \lambda(\eta) + S(f + \varphi_\varepsilon) + \int_{\mathbf{R}^3\setminus\mathcal{O}_\eta} \Psi_{\delta,\varepsilon}(f^n - f)dv + 2C_\varepsilon\|f_\delta - f\|_{L^1_1}, \end{aligned}$$

where

$$\lambda(\eta)=\sup_{n\geqslant 1}\int_{\mathcal{O}_\eta} s(f^n)dv\to 0 \quad (\eta\to 0) \tag{4.9}$$

which is due to the L^1 weak compactness of $\{s(f^n)\}$ (see Lemma 2). Thus first letting $n\to\infty$ (using (4.8)) and then letting $\eta\to 0+$ (using (4.9)) we obtain

$$\limsup_{n\to\infty} S(f^n)\leqslant S(f+\varphi_\varepsilon)+2C_\varepsilon\|f_\delta-f\|_{L^1_1}.$$

Further letting $\delta\to 0+$ gives

$$\limsup_{n\to\infty} S(f^n)\leqslant S(f+\varphi_\varepsilon)\quad (0<\varepsilon\leqslant 1). \tag{4.10}$$

Finally since $0\leqslant s(f+\varphi_\varepsilon)\leqslant s(f+\varphi_1)\in L^1(\mathbf{R}^3)$, it follows from dominated convergence theorem that $S(f+\varphi_\varepsilon)\to S(f)$ as $\varepsilon\to 0+$. Thus in (4.10) letting $\varepsilon\to 0+$ leads to (4.6) and the proof is complete. ∎

The last lemma deals with the Csiszár–Kullback–Pinsker type inequalities (see (4.15), (4.16)) and an inverse inequality (see (4.14)).

Lemma 4. Let $0\leqslant f\in L^1_2(\mathbf{R}^3)$ and $\Omega(v)=ae^{-b|v|^2}/(1-ae^{-b|v|^2})$ with $0<a\leqslant 1$, $0<b<\infty$. Then (1)

$$\|f-\Omega\|_{L^1}\leqslant\|f\|_{L^1}-\|\Omega\|_{L^1}+C_b\left(\|f-\Omega\|_{L^1_2(\backslash 0)}\right)^{1/3}. \tag{4.11}$$

(2) If

$$\int_{\mathbf{R}^3}|v|^2 f(v)dv\leqslant\int_{\mathbf{R}^3}|v|^2\Omega(v)dv \tag{4.12}$$

and either $a=1$ or

$$\int_{\mathbf{R}^3} f(v)dv\leqslant\int_{\mathbf{R}^3}\Omega(v)dv, \tag{4.13}$$

then

$$0\leqslant S(\Omega)-S(f)\leqslant C_b\left(\|f-\Omega\|_{L^1_2(\backslash 0)}\right)^{1/2}, \tag{4.14}$$

$$\|f-\Omega\|_{L^1_2(\backslash 0)}\leqslant C_b\,[S(\Omega)-S(f)]^{1/2}. \tag{4.15}$$

(3) If (4.12) and (4.13) are both satisfied, then

$$\|f-\Omega\|_{L^1_2}\leqslant C_b\,[S(\Omega)-S(f)]^{1/6}\,. \tag{4.16}$$

Here $C_b=C_0\left((1/b)^{1/4}+(1/b)^{9/4}\right)$ and C_0 is an absolute constant.

Proof. Part (1) is based on the identity (using $(y)^+=\max\{y\,,\,0\}$)

$$|f-\Omega|=f-\Omega+2(\Omega-f)^+ \tag{4.17}$$

and a simple estimate (because $0<a\leqslant 1$)

$$\Omega(v)=\frac{1}{\frac{1}{a}e^{b|v|^2}-1}\leqslant\frac{1}{e^{b|v|^2}-1}\leqslant\frac{1}{b|v|^2} \tag{4.18}$$

from which we deduce for any $\delta>0$

$$\begin{aligned}\int_{\mathbf{R}^3}(\Omega-f)^+dv &\leqslant \int_{|v|\leqslant\delta}\Omega dv+\frac{1}{\delta^2}\int_{|v|>\delta}|v|^2(\Omega-f)^+dv\\ &\leqslant \frac{4\pi}{b}\delta+\frac{1}{\delta^2}\|f-\Omega\|_{L^1_2(\backslash 0)}\,.\end{aligned}$$

Minimizing the right hand side with respect to $\delta>0$ gives

$$\int_{\mathbf{R}^3}(\Omega-f)^+dv\leqslant C_0(1/b)^{2/3}\left(\|f-\Omega\|_{L^1_2(\backslash 0)}\right)^{1/3} \tag{4.19}$$

and (4.11) follows from (4.17) and (4.19).

To prove Part (2) we first prove (4.15)–(4.16). For any given $v\in\mathbf{R}^3$ we denote $f=f(v),\,\Omega=\Omega(v)$. Let

$$\Phi_\tau(f,\Omega)=[\Omega+\tau(f-\Omega)][1+\Omega+\tau(f-\Omega)]\,,\quad \tau\in[0,1]\,.$$

Then by (4.1)–(4.2) and Taylor's formula we have

$$0\leqslant\int_0^1\frac{(1-\tau)(f-\Omega)^2}{\Phi_\tau(f,\Omega)}d\tau=s(\Omega)-s(f)+s'(\Omega)(f-\Omega)\,. \tag{4.20}$$

Since

$$s'(\Omega)(f-\Omega)=(\log(1/a))(f-\Omega)+b|v|^2(f-\Omega)$$

and $0 < a \leqslant 1, b > 0$, the assumption in Part (2) implies that

$$\int_{\mathbf{R}^3} s'(\Omega)(f-\Omega)dv \leqslant 0$$

and thus by integrating (4.20) we obtain

$$0 \leqslant \int_{\mathbf{R}^3}\int_0^1 \frac{(1-\tau)(f-\Omega)^2}{\Phi_\tau(f,\Omega)}d\tau\, dv \leqslant S(\Omega)-S(f)\,. \tag{4.21}$$

Next by assumption (4.12) and the identity (4.17) we have

$$\|f-\Omega\|_{L^1_2(\backslash 0)} \leqslant 2\int_{\mathbf{R}^3}|v|^2(\Omega-f)^+dv\,. \tag{4.22}$$

Note that if $v \in \mathbf{R}^3$ satisfies $\Omega(v) > f(v)$, then $0 < \Phi_\tau(f,\Omega) \leqslant \Omega(1+\Omega)$ for all $\tau \in (0,1)$. From this we deduce (by Cauchy–Schwarz inequality)

$$\begin{aligned}(\Omega-f)^+ &= 2\int_0^1 \frac{\sqrt{1-\tau}(\Omega-f)^+}{\sqrt{\Phi_\tau(f,\Omega)}}\cdot\sqrt{(1-\tau)\Phi_\tau(f,\Omega)}\;d\tau\\ &\leqslant 2\sqrt{\int_0^1 \frac{(1-\tau)(\Omega-f)^2}{\Phi_\tau(f,\Omega)}\;d\tau}\cdot\sqrt{\frac{1}{2}\Omega(1+\Omega)}\,.\end{aligned}$$

Multiplying $|v|^2$ to this inequality and applying Cauchy–Schwarz inequality again we obtain (using (4.21))

$$\int_{\mathbf{R}^3}|v|^2(\Omega-f)^+dv \leqslant \sqrt{2}\sqrt{S(\Omega)-S(f)}\sqrt{\int_{\mathbf{R}^3}|v|^4\Omega(1+\Omega)dv}\;. \tag{4.23}$$

Applying inequalities in (4.18) we have

$$|v|^4\Omega(v)(1+\Omega(v)) \leqslant \frac{1}{e^{b|v|^2}-1}\left(|v|^4+\frac{|v|^2}{b}\right)$$

from which we deduce (with an absolute constant C_0)

$$\int_{\mathbf{R}^3}|v|^4\Omega(1+\Omega)dv \leqslant C_0\left((1/b)^{7/2}+(1/b)^{5/2}\right)\,.$$

Therefore by (4.22) and (4.23) we obtain (with a different C_0)

$$\|f-\Omega\|_{L^1_2(\backslash 0)} \leqslant C_0\left((1/b)^{7/4}+(1/b)^{5/4}\right)[S(\Omega)-S(f)]^{1/2}. \qquad (4.24)$$

This proves (4.15). Next assume that (4.12) and (4.13) are both satisfied. Then by (4.17) and (4.19) we have

$$\|f-\Omega\|_{L^1} \leqslant C_0(1/b)^{2/3}\left(\|f-\Omega\|_{L^1_2(\backslash 0)}\right)^{1/3}.$$

This together with

$$\|f-\Omega\|_{L^1_2(\backslash 0)} \leqslant 2\|\Omega\|_{L^1_2(\backslash 0)} \leqslant 2\int_{\mathbf{R}^3}\frac{|v|^2}{e^{b|v|^2}-1}dv = C_0(1/b)^{5/2}$$

gives

$$\begin{aligned}\|f-\Omega\|_{L^1_2} &= \|f-\Omega\|_{L^1}+\|f-\Omega\|_{L^1_2(\backslash 0)}\\ &\leqslant C_0\left((1/b)^{2/3}+(1/b)^{5/3}\right)\left(\|f-\Omega\|_{L^1_2(\backslash 0)}\right)^{1/3} \qquad (4.25)\end{aligned}$$

and thus (4.16) follows from (4.25) and (4.24).

Finally we prove the inverse inequality (4.14). By monotonicity (4.3) and the inequality $0\leqslant s'(y)=\log(1+1/y)\leqslant 4y^{-1/4}$ and noting that $f\geqslant 0$ we have

$$\begin{aligned}S(\Omega)-S(f) &= \int_{\mathbf{R}^3}[s(\Omega)-s(f)]dv \leqslant \int_{\Omega>f}[s(\Omega)-s(f)]dv\\ &= \int_{\Omega>f}\int_0^1 \log\left(1+\frac{1}{f+\tau(\Omega-f)}\right)(\Omega-f)d\tau dv\\ &\leqslant 4\int_{\Omega>f}\int_0^1 \tau^{-1/4}(\Omega-f)^{3/4}d\tau\, dv\\ &= \frac{16}{3}\int_{\mathbf{R}^3}[(\Omega-f)^+]^{3/4}dv. \qquad (4.26)\end{aligned}$$

For any $\delta>0$, applying the inequality $\Omega(v)\leqslant\frac{1}{b|v|^2}$ and the Hölder inequality we have

$$\begin{aligned}&\int_{\mathbf{R}^3}[(\Omega-f)^+]^{3/4}dv\\ &\leqslant \int_{|v|\leqslant\delta}\Omega^{3/4}dv+\int_{|v|>\delta}|v|^{-3/2}\cdot|v|^{3/2}[(\Omega-f)^+]^{3/4}dv\end{aligned}$$

$$\leqslant \int_{|v|\leqslant\delta}\left(\frac{1}{b|v|^2}\right)^{3/4} dv + \left(\int_{|v|>\delta}|v|^{-6}dv\right)^{1/4}\left(\int_{|v|>\delta}|v|^2(\Omega-f)^+dv\right)^{3/4}$$
$$\leqslant C_0 b^{-3/4}\delta^{3/2} + C_0\delta^{-3/4}\left(\|f-\Omega\|_{L^1_2(\backslash 0)}\right)^{3/4}.$$

Minimizing the right-hand with respect to $\delta>0$ gives

$$\int_{\mathbf{R}^3}[(\Omega-f)^+]^{3/4}dv \leqslant C_0\,(1/b)^{1/4}\left(\|f-\Omega\|_{L^1_2(\backslash 0)}\right)^{1/2}$$

which together with (4.26) gives (4.14) and the proof is complete. ∎

5. STRONG CONVERGENCE OF L^1-SEQUENCE

Proposition 3. Let $B(v-v_*,\omega)$ be given in Proposition 2 and assume further that $\underline{b}(\tau)>0$ for $\tau\in(0,1)$. Let $D(f)$ be the entropy dissipation corresponding to $B(v-v_*,\omega)$. Let $\{f_n\}_{n=1}^{\infty}$ be non-negative functions in $L^1_4(\mathbf{R}^3)$ satisfying

$$\sup_{n\geqslant 1}\|f_n\|_{L^1_4}<\infty, \qquad \inf_{n\geqslant 1} S(f_n)>0, \tag{5.1}$$

$$\lim_{n\to\infty}\int_{\mathbf{R}^3} f_n(v)(1,v,|v|^2)dv=(N,0,E), \qquad \lim_{n\to\infty} D(f_n)=0, \tag{5.2}$$

where $0<N,E<\infty$. Let $\Omega_{a,b}$ (with $0<a\leqslant 1, 0<b<\infty$) be the unique Bose–Einstein distribution determined by (N,E). Then (whatever $\overline{T}\geqslant\overline{T}_c$ or $\overline{T}<\overline{T}_c$)

$$\lim_{n\to\infty}\|f_n-\Omega_{a,b}\|_{L^1_2(\backslash 0)}=0, \qquad \lim_{n\to\infty} S(f_n)=S(\Omega_{a,b}). \tag{5.3}$$

Moreover if $\overline{T}\geqslant\overline{T}_c$, then

$$\lim_{n\to\infty}\|f_n-\Omega_{a,b}\|_{L^1_2}=0. \tag{5.4}$$

Proof. **Step 1**. We prove that if ψ_j,ψ_∞ are non-negative functions in $L^1_2(\mathbf{R}^3)$ satisfying $\sup_{j\geqslant 1}\|\psi_j\|_{L^1_2}<\infty$ and

$$\psi_j(v)\to\psi_\infty(v) \quad (j\to\infty) \quad \text{a.e. } v\in\mathbf{R}^3, \tag{5.5}$$

then

$$S(\psi_j) \to S(\psi_\infty) \quad (j \to \infty). \tag{5.6}$$

In fact Lemma 2 and the boundedness of $\{\psi_j\}$ in $L^1_2(\mathbf{R}^3)$ imply that $\{s(\psi_j)\}$ is weakly compact[2] in $L^1(\mathbf{R}^3)$. Thus the pointwise convergence (5.5) implies (5.6).

Step 2. Let $g_n = f_n/(1+f_n)$. We prove that $\{Q^+(g_n, g_n)\}_{n=1}^{\infty}$ is compact in $L^1(\mathbf{R}^3)$. It suffices to prove that

$$\sup_{n\geqslant 1} \|Q^+(g_n, g_n)\|_{L^1_2} < \infty, \tag{5.7}$$

$$\sup_{n\geqslant 1} \|Q^+(g_n, g_n)(\cdot + h) - Q^+(g_n, g_n)\|_{L^1} \to 0 \quad \text{as } h \to 0. \tag{5.8}$$

First of all we have by (5.1) and $0 \leqslant g_n \leqslant f_n$ that

$$\sup_{n\geqslant 1} \|g_n\|_{L^1_4} \leqslant \sup_{n\geqslant 1} \|f_n\|_{L^1_4} < \infty. \tag{5.9}$$

By assumption on B we have $A(|z|) = \int_{\mathbf{S}^2} B(z,\omega)d\omega \leqslant A_0^*$ which gives $L(g_n)(v) \leqslant A_0^* \|g_n\|_{L^1}$ for all $v \in \mathbf{R}^3$ so that

$$\|Q^+(g_n, g_n)\|_{L^1_2} = \|Q^-(g_n, g_n)\|_{L^1_2} = \|g_n L(g_n)\|_{L^1_2} \leqslant A_0^* (\|g_n\|_{L^1_2})^2$$

and thus (5.7) follows from (5.9).

The proof of (5.8) can be reduced to prove the boundedness of $\{Q^+(g_n, g_n)\}$ in the Sobolev space $H^1(\mathbf{R}^3)$ by assuming that the kernel B satisfies some smoothness conditions (see Lions,[18] Bouchut and Desvillttes,[2] Wennebrg[34,35]). And then apply a smooth approximation to the original B. Here we shall use a weaker but enough estimate from ref. 19 without the smoothness assumption: By $A(|z|) \leqslant A_0^*$ we have

$$0 < \int_0^\infty \frac{r^2[A(r)]^2}{(1+r^2)^2} dr \leqslant (A_0^*)^2 \int_0^\infty \frac{r^2}{(1+r^2)^2} dr < \infty$$

which implies that there is a measurable function $\Phi_B(\xi)$ on $\mathbf{R}^3$ that depends only on the kernel B, satisfying $\Phi_B(\xi) \geqslant c > 0$ and $\Phi_B(\xi) \to \infty$ as $|\xi| \to \infty$, such that

$$\int_{\mathbf{R}^3} \Phi_B(\xi) |\mathcal{F}(Q^+(g_n, g_n))(\xi)|^2 d\xi \leqslant \iint_{\mathbf{R}^3 \times \mathbf{R}^3} g_n^2(v) g_n^2(v_*)(1+|v-v_*|^2)^2 dv\, dv_*.$$

[2]In this paper, the "compact" should be understood as "relatively compact".

The right-hand side of this inequality is less than $4(\|g_n\|_{L^1_4})^2$ (because $0 \leqslant g_n \leqslant 1$). Here $\mathcal{F}$ denotes the Fourier form. Therefore by (5.9) we obtain

$$\sup_{n\geqslant 1}\int_{\mathbf{R}^3}\Phi_B(\xi)|\mathcal{F}(Q^+(g_n,g_n))(\xi)|^2 d\xi < \infty .$$

This implies that

$$\Lambda(h) := \sup_{n\geqslant 1}\|Q^+(g_n,g_n)(\cdot+h) - Q^+(g_n,g_n)\|^2_{L^2} \to 0 \quad \text{as } h\to 0 .$$

Now for all $|h|\leqslant 1$ and all $2\leqslant R<\infty$

$$\begin{aligned}
&\|Q^+(g_n,g_n)(\cdot+h) - Q^+(g_n,g_n)\|_{L^1}\\
&\quad\leqslant \int_{|v|\leqslant R}|Q^+(g_n,g_n)(v+h) - Q^+(g_n,g_n)(v)|dv\\
&\qquad + \int_{|v|>R}|Q^+(g_n,g_n)(v+h) - Q^+(g_n,g_n)(v)|dv\\
&\quad\leqslant 3R^{3/2}\sqrt{\Lambda(h)} + \frac{8}{R^2}\int_{|v|>R/2}|v|^2|Q^+(g_n,g_n)(v)|dv .
\end{aligned}$$

Thus by (5.7)

$$\sup_{n\geqslant 1}\|Q^+(g_n,g_n)(\cdot+h) - Q^+(g_n,g_n)\|_{L^1} \leqslant 3R^{3/2}\sqrt{\Lambda(h)} + \frac{C}{R^2},$$

where $C<\infty$ is independent of (R,h). This implies (5.8).

Step 3. Let $M_{a,b} = \Omega_{a,b}/(1+\Omega_{a,b})$. In this step we prove that if

$$\lim_{n\to\infty}\|(1-M_{a,b})(f_n - \Omega_{a,b})\|_{L^1} = 0 \tag{5.10}$$

then (5.3) and (5.4) hold.

Suppose (5.10) holds. Then the L^1_4-boundedness of $\{f_n\}$ implies that (5.10) holds also in $L^1_2(\mathbf{R}^3)$:

$$\lim_{n\to\infty}\|(1-M_{a,b})(f_n - \Omega_{a,b})\|_{L^1_2} = 0 . \tag{5.11}$$

Next by $0<a\leqslant 1$ we have

$$1 - M_{a,b}(v) \geqslant 1 - e^{-b|v|^2} \geqslant \frac{b|v|^2}{1+b|v|^2}$$

which implies (with $C=(1+b)/b$)

$$|v|^2 \leqslant C(1+|v|^2)(1-M_{a,b}(v)) \tag{5.12}$$

and so as $n\to\infty$

$$\|f_n-\Omega_{a,b}\|_{L^1_2(\setminus 0)} \leqslant C\|(1-M_{a,b})(f_n-\Omega_{a,b})\|_{L^1_2} \to 0\,. \tag{5.13}$$

Choose a subsequence $\{n_k\}_{k=1}^{\infty}\subset\{n\}$ such that

$$\lim_{k\to\infty}|S(f_{n_k})-S(\Omega_{a,b})| = \limsup_{n\to\infty}|S(f_n)-S(\Omega_{a,b})|\,. \tag{5.14}$$

By (5.13) there exists a subsequence $\{n_{k_j}\}_{j=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ such that $f_{n_{k_j}}(v)\to \Omega_{a,b}(v)$ $(j\to\infty)$ a.e. $v\in\mathbf{R}^3$ so that by Step 1 we have $S(f_{n_{k_j}})\to S(\Omega_{a,b})$ as $j\to\infty$. This together with (5.14) implies $\limsup_{n\to\infty}|S(f_n)-S(\Omega_{a,b})|=0$ and (5.3) is proven.

Now assume that $\overline{T}\geqslant\overline{T}_c$. Then

$$\lim_{n\to\infty}\|f_n\|_{L^1}=N=\|\Omega_{a,b}\|_{L^1}$$

and thus applying (4.11) in Part (1) of Lemma 4 together with (5.13) we obtain (5.4).

Step 4. We now prove that (5.10) holds true. Let $\{n_j\}_{j=1}^{\infty}$ be a subsequence of $\{n\}$ such that

$$\lim_{j\to\infty}\|(1-M_{a,b})(f_{n_j}-\Omega_{a,b})\|_{L^1} = \limsup_{n\to\infty}\|(1-M_{a,b})(f_n-\Omega_{a,b})\|_{L^1}. \tag{5.15}$$

Recall that

$$g_{n_j}(v)=\frac{f_{n_j}(v)}{1+f_{n_j}(v)}\,.$$

We shall prove that a subsequence of $\{g_{n_j}\}_{j=1}^{\infty}$ converges to a Maxwellian. First of all the L^{∞} bounds $0\leqslant g_{n_j}\leqslant 1$ and (5.9) imply that $\{g_{n_j}\}_{j=1}^{\infty}$ is weakly compact in $L^1(\mathbf{R}^3)$. So there exist a subsequence of $\{n_j\}_{j=1}^{\infty}$, still denote as $\{n_j\}_{j=1}^{\infty}$, and a function $g_\infty\in L^1(\mathbf{R}^3)$ such that

$$\int_{\mathbf{R}^3}\psi g_{n_j}\,dv\to\int_{\mathbf{R}^3}\psi g_\infty\,dv \quad (j\to\infty) \quad \forall\,\psi\in L^{\infty}(\mathbf{R}^3)\,. \tag{5.16}$$

This implies that, after modification on a null set, $0 \leqslant g_\infty(v) \leqslant 1$ for all $v \in \mathbf{R}^3$. In the following, further extraction of subsequence of $\{g_{n_j}\}_{j=1}^{\infty}$ may be needed. For notational convenience (without loss of generality) we denote $\{g_{n_j}\}_{j=1}^{\infty}$ to be various convergent subsequence of $\{g_{n_j}\}_{j=1}^{\infty}$. We claim that $\int_{\mathbf{R}^3} g_\infty dv > 0$. Otherwise, $\int_{\mathbf{R}^3} g_\infty dv = 0$, then (5.16) implies $\lim_{j\to\infty} \|g_{n_j}\|_{L^1} = 0$ so that $g_{n_j}(v) \to 0 (j \to \infty)$ a.e. $v \in \mathbf{R}^3$. This implies that

$$f_{n_j}(v) = \frac{g_{n_j}(v)}{1 - g_{n_j}(v)} \to 0 \quad (j \to \infty) \text{ a.e. } v \in \mathbf{R}^3 .$$

By Step 1, this implies $\lim_{j\to\infty} S(f_{n_j}) = S(0) = 0$ which contradicts the assumption $\inf_{n\geqslant 1} S(f_n) > 0$. Therefore $\int_{\mathbf{R}^3} g_\infty dv > 0$. By positivity $A(z) > 0$ for all $z \neq 0$, this implies that $L(g_\infty)(v) > 0$ for all $v \in \mathbf{R}^3$. Next by L^1-compactness of $\{Q^+(g_n, g_n)\}$ there exists a function $0 \leqslant G \in L^1(\mathbf{R}^3)$, such that

$$\lim_{j\to\infty} \|Q^+(g_{n_j}, g_{n_j}) - G\|_{L^1} = 0$$

which implies that

$$\lim_{j\to\infty} Q^+(g_{n_j}, g_{n_j})(v) = G(v) \text{ a.e. } v \in \mathbf{R}^3 .$$

Let us write (recalling (1.13))

$$g_{n_j}(v) = \frac{Q^-(g_{n_j}, g_{n_j})(v)}{L(g_{n_j})(v)} = \frac{Q^+(g_{n_j}, g_{n_j})(v)}{L(g_{n_j})(v)} - \frac{Q(g_{n_j}, g_{n_j})(v)}{L(g_{n_j})(v)} . \quad (5.17)$$

Since $A(z) \leqslant A_0^*$ on $\mathbf{R}^3$, it follows from (5.16) that for all $v \in \mathbf{R}^3$

$$\lim_{j\to\infty} L(g_{n_j})(v) = \lim_{j\to\infty} \int_{\mathbf{R}^3} g_{n_j}(v_*) A(v - v_*) dv_* = L(g_\infty)(v) > 0 .$$

On the other hand, the assumption $\lim_{n\to\infty} D(f_n) = 0$ and Proposition 2 imply that

$$\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\, |g_n' g_{n*}' - g_n g_{n*}| d\omega\, dv_*\, dv \leqslant \sqrt{3A_0}\, \|f_n\|_{L^1} \sqrt{D(f_n)} \to 0 \quad (5.18)$$

as $n \to \infty$. This implies $\lim_{n\to\infty} \|Q(g_n, g_n)\|_{L^1} = 0$ and thus

$$Q(g_{n_j}, g_{n_j})(v) \to 0 \quad (j \to \infty) \text{ a.e. } v \in \mathbf{R}^3 .$$

Therefore from (5.17) we obtain the pointwise convergence:

$$g_{n_j}(v) \to \frac{G(v)}{L(g_\infty)(v)} =: \tilde{g}_\infty(v) \quad (j \to \infty) \text{ a.e. } v \in \mathbf{R}^3 .$$

Since $\{g_{n_j}\}_{j=1}^\infty$ is weakly compact in $L^1(\mathbf{R}^3)$, it follows that $\lim_{j\to\infty} \|g_{n_j} - \tilde{g}_\infty\|_{L^1} = 0$. By (5.16), this implies $\tilde{g}_\infty = g_\infty$ a.e. on $\mathbf{R}^3$ and so $\lim_{j\to\infty} \|g_{n_j} - g_\infty\|_{L^1} = 0$. Now applying Fatou's Lemma and (5.18) we have

$$\begin{aligned} &\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\,|g'_\infty g'_{\infty *} - g_\infty g_{\infty *}|\,d\omega\, dv\, dv_* \\ &\quad \leqslant \liminf_{j\to\infty} \iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} B\,|g_{n_j}{}' g_{n_j}{}'_* - g_{n_j} g_{n_j *}|\,d\omega\, dv\, dv_* = 0 . \end{aligned}$$

Since $B(z,\omega) > 0$ for a.e. $(z,\omega) \in \mathbf{R}^3 \times \mathbf{S}^2$ and $0 \leqslant g_\infty \in L^1(\mathbf{R}^3)$, it follows that $g'_\infty g'_{\infty *} = g_\infty g_{\infty *}$ for a.e. $(v, v_*, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$ and hence g_∞ is a Maxwellian:

$$g_\infty(v) = M(v) = \tilde{a} e^{-\tilde{b}|v-v_0|^2} \;\text{ a.e. } v \in \mathbf{R}^3 .$$

Therefore

$$\|g_{n_j} - M\|_{L^1} \to 0 \text{ as } j \to \infty . \tag{5.19}$$

Since $0 \leqslant g_\infty \leqslant 1$ and $\int_{\mathbf{R}^3} g_\infty dv > 0$, these imply that $0 < \tilde{a} \leqslant 1\,, 0 < \tilde{b} < \infty$. Let

$$\Omega(v) = \frac{M(v)}{1 - M(v)} = \frac{\tilde{a} e^{-\tilde{b}|v-v_0|^2}}{1 - \tilde{a} e^{-\tilde{b}|v-v_0|^2}} .$$

Then by Proposition 2 and (5.19) we have

$$\|(1-M)(f_{n_j} - \Omega)\|_{L^1} \leqslant C\left(\|g_{n_j} - M\|_{L^1} + \sqrt{D(f_{n_j})}\right) \to 0 \quad (j \to \infty) . \tag{5.20}$$

This convergence together with the L^1_4 bounds $\sup_{n\geqslant 1} \|f_n\|_{L^1_4} < \infty$ imply

$$\|(1-M)(f_{n_j} - \Omega)\|_{L^1_2} \to 0 \quad (j \to \infty) . \tag{5.21}$$

We need to prove $\Omega=\Omega_{a,b}$, i.e. $v_0=0, \tilde{a}=a$, and $\tilde{b}=b$. If this is done, then (5.15) gives (5.10) and the proof of the proposition is complete by Step 3. We first prove $v_0=0$. As shown in (5.12) we have (with C denoting different constants) $|v-v_0|^2\leqslant C(1+|v-v_0|^2)(1-M(v))\leqslant C(1+|v|^2)(1-M(v))$. This implies

$$\int_{\mathbf{R}^3}|v-v_0|^2|f_{n_j}(v)-\Omega(v)|dv\leqslant C\|(1-M)(f_{n_j}-\Omega)\|_{L^1_2}\to 0 \quad (5.22)$$

as $j\to\infty$. By Cauchy–Schwarz inequality, this gives that as $j\to\infty$

$$\left|\int_{\mathbf{R}^3}(v-v_0)[f_{n_j}(v)-\Omega(v)]dv\right|\leqslant C\left(\int_{\mathbf{R}^3}|v-v_0|^2|f_{n_j}(v)-\Omega(v)|dv\right)^{1/2}\to 0\,.$$

Since $\int_{\mathbf{R}^3}(v-v_0)\Omega(v)dv=0, \int_{\mathbf{R}^3}vf_n(v)dv\to 0, \int_{\mathbf{R}^3}f_n(v)dv\to N>0$ $(n\to\infty)$, it follows that $v_0=0$. Therefore we can write

$$\Omega(v)=\Omega_{\tilde{a},\tilde{b}}(v),\qquad M(v)=M_{\tilde{a},\tilde{b}}(v)\,.$$

Applying (5.22) again and using Fatou Lemma we obtain

$$\int_{\mathbf{R}^3}|v|^2\Omega_{\tilde{a},\tilde{b}}(v)dv=\lim_{j\to\infty}\int_{\mathbf{R}^3}|v|^2f_{n_j}(v)dv=E\,, \quad (5.23)$$

$$\int_{\mathbf{R}^3}\Omega_{\tilde{a},\tilde{b}}(v)dv\leqslant\lim_{j\to\infty}\int_{\mathbf{R}^3}f_{n_j}(v)dv=N\,. \quad (5.24)$$

To prove $\tilde{a}=a$ and $\tilde{b}=b$ we first consider the case $0<\tilde{a}<1$. For this case we have $1-M_{\tilde{a},\tilde{b}}(v)\geqslant 1-\tilde{a}>0$ for all $v\in\mathbf{R}^3$ and thus

$$\|f_{n_j}-\Omega_{\tilde{a},\tilde{b}}\|_{L^1_2}\leqslant\frac{1}{1-\tilde{a}}\|(1-M_{\tilde{a},\tilde{b}})(f_{n_j}-\Omega_{\tilde{a},\tilde{b}})\|_{L^1_2}\to 0\quad (j\to\infty)\,.$$

This implies $\int_{\mathbf{R}^3}\Omega_{\tilde{a},\tilde{b}}(v)dv=N$ which together with (5.23) implies that $\Omega_{\tilde{a},\tilde{b}}$ arrives at the conditional maximum (2.3) and hence by definition and the uniqueness of $\Omega_{a,b}$ we conclude that $\tilde{a}=a$ and $\tilde{b}=b$.

Next suppose $\tilde{a}=1$. If $\int_{\mathbf{R}^3}\Omega_{1,\tilde{b}}(v)dv=N$, then, as shown above, $\tilde{a}=a(=1)$ and $\tilde{b}=b$. If $\int_{\mathbf{R}^3}\Omega_{1,\tilde{b}}(v)dv\neq N$, then (5.24) implies $\int_{\mathbf{R}^3}\Omega_{1,\tilde{b}}(v)dv<N$ so that by (2.7) (with $\tilde{a}=1$) we have

$$1>\frac{1}{N}\int_{\mathbf{R}^3}\Omega_{1,\tilde{b}}(v)dv=(\overline{T}/\overline{T}_c)^{3/5}$$

i.e. $\overline{T} < \overline{T}_c$ which implies by Proposition 1 that $a = 1$ and therefore $\tilde{b} = b$ because from (5.23), (2.6) (with $\tilde{a} = 1$) and Proposition 1 one sees that $\tilde{b}$ and b have the same expression in terms of E.

We have proved that $\Omega = \Omega_{a,b}$, $M = M_{a,b}$. Therefore by (5.20) we obtain

$$\lim_{j\to\infty} \|(1 - M_{a,b})(f_{n_j} - \Omega_{a,b})\|_{L^1} = 0$$

which together with (5.15) implies (5.10) and the proof is complete. ∎

6. STRONG CONVERGENCE OF L^1-SOLUTIONS

In this section we consider hard potentials with certain cutoffs that insure the global existence, uniqueness and moment production property of conservative L^1 isotropic solutions of Eq. (BBE).[20] We will prove that the convergence of the solutions (as $t \to \infty$) is uniform with respect to the cutoff constants.

Theorem 1. Let $B(v - v_*, \omega) = b(\cos\theta)|v - v_*|^\gamma$ where $0 < \gamma \leqslant 1$ and $b(\cdot) \geqslant 0$ is a bounded Borel function on [0, 1] satisfying $b(\tau) > 0$ for all $\tau \in (0, 1)$. Given any set $\mathcal{K} \subset [1, \infty)$. For any $K \in \mathcal{K}$, let

$$B_K(v - v_*, \omega) = \min\{B(v - v_*, \omega)\,,\ K|v - v_*|^3 \cos^2\theta\, \sin\theta\} \tag{6.1}$$

and let $\{f_0^K\}_{K\in\mathcal{K}} \subset L^1_2(\mathbf{R}^3)$ be a family of non-negative isotropic initial data satisfying

$$\inf\nolimits_{K\in\mathcal{K}} S(f_0^K) > 0, \tag{6.2}$$

$$\int_{\mathbf{R}^3} (1, v, |v|^2) f_0^K(v)dv = (N, 0, E) \quad \forall K \in \mathcal{K}, \tag{6.3}$$

where $N, E > 0$ are constants independent of K. For every $K \in \mathcal{K}$, let $0 \leqslant f^K \in C^1([0, \infty); L^1(\mathbf{R}^3))$ be the unique conservative isotropic solution of Eq. (BBE) with the kernel $B_K(v - v_*, \omega)$ satisfying $f^K|_{t=0} = f_0^K$. Let $\overline{T}, \overline{T}_c$ and $\Omega_{a,b}$ be the kinetic temperatures and the unique Bose–Einstein distribution determined by N, E. Then

(I) For all $K \in \mathcal{K}$

$$S(f^K(t)) = S(f_0^K) + \frac{1}{4}\int_0^t D_K(f^K(\tau))d\tau \quad \forall t \geqslant 0, \tag{6.4}$$

$$\sup_{K\in\mathcal{K}} \|f^K(t)\|_{L^1_s} \leqslant C_s(1 + 1/t)^{(s-2)/\gamma} \quad \forall t > 0, \quad \forall s > 2, \tag{6.5}$$

where $D_K(f)$ is the entropy dissipation corresponding to $B_K(v-v_*,\omega)$, the constants C_s depend only on $b(\cdot), \gamma, N, E$ and s.

(II.1) For any temperature ($\overline{T} \geqslant \overline{T}_c$ or $\overline{T} < \overline{T}_c$) and for all $K \in \mathcal{K}$,

$$0 \leqslant S(\Omega_{a,b}) - S(f^K(t)) \leqslant C_b \left(\| f^K(t) - \Omega_{a,b} \|_{L^1_2(\setminus 0)} \right)^{1/2} \quad \forall t \geqslant 0, \tag{6.6}$$

$$\| f^K(t) - \Omega_{a,b} \|_{L^1_2(\setminus 0)} \leqslant C_b \left[S(\Omega_{a,b}) - S(f^K(t)) \right]^{1/2} \quad \forall t \geqslant 0, \tag{6.7}$$

and

$$\sup_{K \in \mathcal{K}} [S(\Omega_{a,b}) - S(f^K(t))] \to 0 \quad (t \to \infty). \tag{6.8}$$

(II.2) If $\overline{T} \geqslant \overline{T}_c$, then for all $K \in \mathcal{K}$

$$\| f^K(t) - \Omega_{a,b} \|_{L^1_2} \leqslant C_b \left[S(\Omega_{a,b}) - S(f^K(t)) \right]^{1/6} \quad \forall t \geqslant 0. \tag{6.9}$$

Here, in (II.1)–(II.2), $C_b = C_0((1/b)^{1/4} + (1/b)^{9/4})$ is the constant given in Lemma 4. In particular, C_b is independent of K.

(II.3) If $\overline{T} < \overline{T}_c$, then there exists $0 < t_0 < \infty$ which is independent of K such that

$$1 - (\overline{T}/\overline{T}_c)^{3/5} + C\delta(t) \geqslant \frac{1}{N} \int_{|v| \leqslant \delta(t)} f^K(v,t)\,dv \geqslant 1 - (\overline{T}/\overline{T}_c)^{3/5} \quad \forall t \geqslant t_0 \tag{6.10}$$

where $C = 5\pi/(bN)$ and

$$\delta(t) := \sup_{K \in \mathcal{K}} \left(\| f^K(t) - \Omega_{1,b} \|_{L^1_2(\setminus 0)} \right)^{1/4} \to 0 \quad (t \to \infty).$$

Proof. For Part (I), the entropy identity (6.4) is a result of ref. 20. The moment production (6.5) has been proven in the proof of Theorem 4 in ref. 22. It should be noted that in ref. 22 we also required that $b(\cdot)$ is continuous and $b(0) = 0$, but these conditions are only used to prove a weak stability of $\{f^K\}_{K \in \mathcal{K}}$. In fact from the proof of Theorem 4 in ref. 22 one can easily check that at least for the cutoff kernels $B_K(v-v_*,\omega)$ the present conditions on $b(\cdot)$ are enough to obtain the moment production property (6.5). Of course for the hard sphere model, i.e. for $b(\tau) = \text{const.}\tau$ and $\gamma = 1$, all requirements are satisfied.

For Part (II), we first prove the entropy inequalities (6.6), (6.7) and (6.9). By definition of $\Omega_{a,b}$ and that f^K conserve the mass and energy, we have for any $K \in \mathcal{K}$

$$\int_{\mathbf{R}^3} |v|^2 f^K(v,t)dv = E = \int_{\mathbf{R}^3} |v|^2 \Omega_{a,b}(v)dv \quad \forall t \geqslant 0.$$

Also by Proposition 1 one sees that the inequality $\overline{T} < \overline{T}_c$ implies $a=1$, while the inequality $\overline{T} \geqslant \overline{T}_c$ implies

$$\int_{\mathbf{R}^3} f^K(v,t)dv = N = \int_{\mathbf{R}^3} \Omega_{a,b}(v)dv \quad \forall t \geqslant 0.$$

Therefore the estimates (4.14)–(4.16) in Lemma 4 apply to deduce the entropy inequalities (6.6), (6.7) and (6.9).

Now we will use Proposition 3 to prove the uniform entropy convergence (6.8). Consider the "minimal" kernel: $\underline{b}(\cos\theta) = \min\{b(\cos\theta), 1\}$,

$$B_{\min}(v-v_*, \omega) = \underline{b}(\cos\theta)\cos^3\theta \,\sin^3\theta \,\min\{|v-v_*|^3, 1\}.$$

Recalling (6.1) and $\mathcal{K} \subset [1,\infty)$ we have

$$B_{\min}(v-v_*,\omega) \leqslant B_K(v-v_*,\omega) \quad \forall K \in \mathcal{K}. \tag{6.11}$$

Let $D_{\min}(f)$ and $D_K(f)$ be the entropy dissipations corresponding to the kernels $B_{\min}(v-v_*,\omega)$ and $B_K(v-v_*,\omega)$ respectively. By (6.11) we have $D_{\min}(f) \leqslant D_K(f)$ for $f \geqslant 0$. To prove (6.8), we choose a sequence $\{t_n\} \subset [2,\infty)$ satisfying $t_n \to \infty$ $(n \to \infty)$ and a sequence $\{K_n\} \subset \mathcal{K}$ such that

$$\lim_{n\to\infty} [S(\Omega_{a,b}) - S(f^{K_n}(t_n))] = \limsup_{t\to\infty} \sup_{K\in\mathcal{K}} [S(\Omega_{a,b}) - S(f^K(t))]. \tag{6.12}$$

By entropy identity (6.4) we have (because $0 \leqslant S(f^K(t)) \leqslant S(\Omega_{a,b})$)

$$\frac{2}{t_n}\int_{t_n/2}^{t_n} D_{K_n}(f^{K_n}(\tau))d\tau = \frac{8}{t_n}\Big(S(f^{K_n}(t_n)) - S(f^{K_n}(t_n/2))\Big) \leqslant \frac{8S(\Omega_{a,b})}{t_n}.$$

This implies that there exist $\tau_n \in [t_n/2, t_n]$ $(n = 1, 2, ...)$ such that

$$D_{K_n}(f^{K_n}(\tau_n)) \leqslant \frac{8S(\Omega_{a,b})}{t_n} \to 0 \quad (n\to\infty).$$

Since $0 \leqslant D_{\min}(f^K(t)) \leqslant D_K(f^K(t))$, it follows that

$$D_{\min}(f^{K_n}(\tau_n)) \to 0 \quad (n \to \infty).$$

Also, by conservation laws, moment production (6.5), $S(f^K(t)) \geqslant S(f_0^K)$, and (6.2)–(6.3) we have

$$\int_{\mathbf{R}^3} (1, v, |v|^2) f^{K_n}(v, \tau_n) dv = (N, 0, E), \quad n = 1, 2, 3, \dots,$$
$$\sup_{n \geqslant 1} \| f^{K_n}(\tau_n) \|_{L^1_4} < \infty, \quad \inf_{n \geqslant 1} S(f^{K_n}(\tau_n)) \geqslant \inf_{K \in \mathcal{K}} S(f_0^K) > 0.$$

Therefore applying Proposition 3 to the kernel $B_{\min}$ with the entropy dissipation $D_{\min}$ and the functions $f_n(v) = f^{K_n}(v, \tau_n)$, and noting that $S(f^{K_n}(\tau_n)) \leqslant S(f^{K_n}(t_n)) \leqslant S(\Omega_{a,b})$ we obtain

$$0 \leqslant S(\Omega_{a,b}) - S(f^{K_n}(t_n)) \leqslant S(\Omega_{a,b}) - S(f^{K_n}(\tau_n)) \to 0 \ (n \to \infty)$$

and therefore (6.8) follows from (6.12). This proves (II.1)–(II.2).

Finally we prove (II.3). Suppose $\overline{T} < \overline{T}_c$. Then $a = 1$ and $\int_{\mathbf{R}^3} \Omega_{1,b}(v) dv < N$ which implies that $(\delta(t))^4 = \sup_{K \in \mathcal{K}} \| f^K(t) - \Omega_{1,b} \|_{L^1_2(\setminus 0)} > 0$ for all $t \geqslant 0$. Let $N_0 = N - \int_{\mathbf{R}^3} \Omega_{1,b} dv$. We compute

$$\int_{|v| \leqslant \delta(t)} f^K(v, t) dv = N_0 + I_K(t), \tag{6.13}$$

$$I_K(t) = \int_{|v| \leqslant \delta(t)} \Omega_{1,b}(v) dv + \int_{|v| > \delta(t)} [\Omega_{1,b}(v) - f^K(v, t)] dv, \tag{6.14}$$

$$\left| \int_{|v| > \delta(t)} [\Omega_{1,b}(v) - f^K(v, t)] dv \right| \leqslant \frac{1}{\delta^2(t)} \| f^K(t) - \Omega_{1,b} \|_{L^1_2(\setminus 0)} \leqslant \delta^2(t). \tag{6.15}$$

Since $\delta(t) \to 0$, there exists $0 < t_0 < \infty$ such that $\delta(t) < \min\{\sqrt{1/b}, \pi/b\}$ for all $t \geqslant t_0$. By inequality $1 + x \leqslant e^x \leqslant 1 + 2x$ for $0 \leqslant x \leqslant 1$ we have for all $t \geqslant t_0$

$$\frac{2\pi}{b} \delta(t) \leqslant \int_{|v| \leqslant \delta(t)} \Omega_{1,b}(v) dv = 4\pi \int_0^{\delta(t)} \frac{r^2}{e^{br^2} - 1} dr \leqslant \frac{4\pi}{b} \delta(t).$$

By (6.14)–(6.15), the left-hand side inequality gives for all $K \in \mathcal{K}$

$$I_K(t) \geqslant \frac{2\pi}{b} \delta(t) - \delta^2(t) = \delta(t) \left(\frac{2\pi}{b} - \delta(t) \right) > 0 \quad \forall t \geqslant t_0,$$

while the right-hand side inequality gives

$$I_K(t) \leqslant \frac{4\pi}{b}\delta(t) + \delta^2(t) \leqslant \frac{5\pi}{b}\delta(t)\,.$$

Combining these with (6.13) and recalling that $N_0 = N(1-(\overline{T}/\overline{T}_c)^{3/5})$ (see (2.10)) we obtain (6.10). ∎

Remark. Theorem 1 implies that at low temperature $\overline{T} < \overline{T}_c$ the solutions $f^K(v,t)$ can be split as a sum of "regular" parts and "singular" parts: $f^K(v,t) = f^K_r(v,t) + f^K_s(v,t)$ where

$$f^K_r(v,t) = f^K(v,t)1_{\{|v|>\delta(t)\}}\,, \quad f^K_s(v,t) = f^K(v,t)1_{\{|v|\leqslant\delta(t)\}}\,.$$

As time goes to infinity, the "regular" parts $f^K_r(v,t)$ converge in $L^1(\mathbf{R}^3)$ to $\Omega_{1,b}$ uniformly in K, while the "singular" parts $f^K_s(v,t)$ converge uniformly to a unique Dirac delta function. That is,

$$\lim_{t\to\infty}\sup_{K\in\mathcal{K}}\int_{\mathbf{R}^3}|f^K_r(v,t) - \Omega_{1,b}(v)|dv = 0\,,$$

$$\lim_{t\to\infty}\sup_{K\in\mathcal{K}}\left|\int_{\mathbf{R}^3} f^K_s(v,t)\varphi(v)dv - N_0\varphi(0)\right| = 0 \quad \forall\varphi\in C\cap L^\infty(\mathbf{R}^3).$$

7. STRONG CONVERGENCE OF DISTRIBUTIONAL SOLUTIONS

In this section we shall use the result of uniformly strong convergence (Theorem 1) to prove the strong convergence to equilibrium and the velocity concentration for distributional solutions of Eq. (BBE) where the collision kernels include the hard sphere model.

In view of existence we have so far only isotropic solutions. As seen in the previous section that in our proofs of the strong convergence and velocity concentration, the isotropic condition (on initial data) is only used to insure the existence of conservative solutions that satisfy the entropy identity and the moment production. Once these hold true, the isotropic condition will never be used in our proofs. Therefore it is benefit to write the isotropic version as a general version. Let $\bar{F}$ be a finite positive Borel measure on $\mathbf{R}_+ \equiv [0,\infty)$. By Riesz representation theorem, there exists a unique finite positive Borel measure F on $\mathbf{R}^3$ such that

$$\int_{\mathbf{R}^3}\psi(v)dF(v) = \int_{\mathbf{R}_+}\left(\frac{1}{4\pi}\int_{\mathbf{S}^2}\psi(r\omega)d\omega\right)d\bar{F}(r) \quad \forall\psi\in C_0(\mathbf{R}^3)\,. \tag{7.1}$$

Also, by Lebesgue–Radon–Nikodym theorem, there exist a unique $0 \leqslant \bar{f} \in L^1(\mathbf{R}_+)$, a unique finite positive Borel measure $\bar{\mu}$ on $\mathbf{R}_+$, and a Borel set $Z \subset \mathbf{R}_+$ satisfying $mes(Z) = 0$, $\bar{\mu}(\mathbf{R}_+ \setminus Z) = 0$, such that $d\bar{F}(r) = \bar{f}(r)dr + d\bar{\mu}(r)$. Let μ be the unique finite positive Borel measure on $\mathbf{R}^3$ determined by $\bar{\mu}$ through the relation (7.1) and let $f(v) = \frac{1}{4\pi|v|^2}\bar{f}(|v|)$. Then

$$dF(v) = f(v)dv + d\mu(v) \tag{7.2}$$

$0 \leqslant f \in L^1(\mathbf{R}^3)$ and μ is the singular part of F. By the uniqueness of the Lebesgue decomposition, (f, μ) is uniquely determined by F (or equivalently by $\bar{F}$) through (7.1)–(7.2). Because of these correspondence, we will not distinguish between $\bar{F}$ and F, and we call the measure F (or $dF(v)$) an isotropic distribution on $\mathbf{R}^3$.

In order to introduce distributional solutions of Eq.(BBE) for the kernel $B(v - v_*, \omega) = b(\cos\theta)|v - v_*|^\gamma$ with $0 \leqslant \gamma \leqslant 1$ we assume that $b(\tau)$ satisfies

$$0 \leqslant b(\cdot) \in C([0,1])\,, \quad \int_0^1 b(\tau)d\tau > 0\,, \quad b(0) = 0\,. \tag{7.3}$$

The condition $b(0) = 0$ is only used in a proof of a weak stability and thus the existence of distributional solutions. Roughly speaking, if $b(0) \neq 0$, then the smooth test functions multiplied by the collision kernel will not be always continuous (see ref. 22). Note that the hard sphere model (1.3) satisfies (7.3).

Distributional Solutions of Eq. (BBE) ([**22**]): Let $b(\cdot)$ satisfy (7.3), $B(v - v_*, \omega) = b(\cos\theta)|v - v_*|^\gamma$ with $0 \leqslant \gamma \leqslant 1$. Let $F_0, F_t \in \mathcal{B}_2^+(\mathbf{R}^3)$ be isotropic distributions for all $t \geqslant 0$. If F_t satisfies the following (i)–(iv), then we say that F_t is a conservative isotropic distributional solution of Eq. (BBE) with the initial datum F_0.

(i) $F_t|_{t=0} = F_0$,

(ii) $\int_{\mathbf{R}^3}(1, v, |v|^2)dF_t(v) = \int_{\mathbf{R}^3}(1, v, |v|^2)dF_0(v) \quad \forall t \geqslant 0$,

(iii) the function $t \mapsto \int_{\mathbf{R}^3}\varphi(|v|^2)dF_t(v)$ is in $C^1(\mathbf{R}_+)$ for all $\varphi(r) \in C_b^2(\mathbf{R}_+)$ (i.e. $\frac{d^k}{dr^k}\varphi(r)$ for $k = 0, 1, 2$ are all continuous and bounded on $\mathbf{R}_+$) and

(iv) the equation

$$\frac{d}{dt}\int_{\mathbf{R}^3}\varphi(|v|^2)dF_t(v)=\langle Q_B(F_t),\varphi\rangle$$

holds for all $t\geqslant 0$ and all $\varphi(r)\in C_b^2(\mathbf{R}_+)$.

Here $\varphi\mapsto\langle Q_B(F_t),\varphi\rangle$ (for every fixed t) is a bounded linear functional constructed in ref. 22. In the following we do not use this functional. What we shall use is only the existence of the distributional solution which is obtained by weak convergence of certain approximate L^1 solutions.

Theorem 2. Let $B(v-v_*,\omega)=b(\cos\theta)|v-v_*|^\gamma$ satisfy $b(\cdot)\in C([0,1])$, $b(\tau)>0$ in $\tau\in(0,1)$, $b(0)=0$ and $0<\gamma\leqslant 1$. Let $F_0\in\mathcal{B}_2^+(\mathbf{R}^3)$ be an isotropic distribution satisfying

$$dF_0(v)=f_0(v)dv+d\mu_0(v),$$

where $0\leqslant f_0\in L_2^1(\mathbf{R}^3)$ is an isotropic function with $\int_{\mathbf{R}^3}f_0(v)dv>0$ and $\mu_0\in\mathcal{B}_2^+(\mathbf{R}^3)$ is the singular part of F_0. Let $N=\int_{\mathbf{R}^3}dF_0(v)$, $E=\int_{\mathbf{R}^3}|v|^2dF_0(v)$, and let $\overline{T}$, $\overline{T}_c$, and $\Omega_{a,b}$ be the temperatures and the Bose–Einstein distribution corresponding to N and E. Then there exists a conservative isotropic distributional solution F_t of Eq. (BBE) with the kernel B and $F_t|_{t=0}=F_0$, such that for the Lebesgue decomposition

$$dF_t(v)=f_t(v)dv+d\mu_t(v),$$

where $0\leqslant f_t\in L_2^1(\mathbf{R}^3)$ and $\mu_t\in\mathcal{B}_2^+(\mathbf{R}^3)$ is the singular part of F_t, the following (I) and (II.1)–(II.3) hold:

(I) F_t is a weak limit of a subsequence $\{f^{n_j}(\cdot,t)\}_{j=1}^\infty$ of L^1 approximate solutions $\{f^n(\cdot,t)\}_{n=1}^\infty$ of Eq. (BBE) with the cutoff kernels (6.1) for $K=n\in\mathcal{K}=\{1,2,3,\dots\}$. Here the weak limit means

$$\lim_{j\to\infty}\int_{\mathbf{R}^3}\varphi(v)f^{n_j}(v,t)dv=\int_{\mathbf{R}^3}\varphi(v)dF_t(v)\quad\forall\varphi\in C_c(\mathbf{R}^3),\ \forall t\geqslant 0. \tag{7.4}$$

(II.1) For any temperature ($\overline{T}\geqslant\overline{T}_c$ or $\overline{T}<\overline{T}_c$) and for all $t\geqslant 0$

$$0\leqslant S(\Omega_{a,b})-S(f_t)\leqslant C_b\left(\|f_t-\Omega_{a,b}\|_{L_2^1(\setminus 0)}\right)^{1/2}, \tag{7.5}$$

$$\int_{\mathbf{R}^3}|v|^2d\mu_t(v)\leqslant\|f_t-\Omega_{a,b}\|_{L_2^1(\setminus 0)}\leqslant C_b[S(\Omega_{a,b})-S(f_t)]^{1/2}, \tag{7.6}$$

and

$$S(f_t) \to S(\Omega_{a,b}) \quad (t \to \infty)\,. \tag{7.7}$$

(II.2) If $\overline{T} \geqslant \overline{T}_c$, then for all $t \geqslant 0$

$$\int_{\mathbf{R}^3} (1+|v|^2)d\mu_t(v) \leqslant \|f_t - \Omega_{a,b}\|_{L^1_2} \leqslant C_b[S(\Omega_{a,b}) - S(f_t)]^{1/6}\,. \tag{7.8}$$

Here, in (II.1)–(II.2), $C_b = C_0((1/b)^{1/4} + (1/b)^{9/4})$ is the constant given in Lemma 4.

(II.3) If $\overline{T} < \overline{T}_c$, then there exists $0 < t_0 < \infty$ such that for all $t \geqslant t_0$

$$1 - (\overline{T}/\overline{T}_c)^{3/5} + C\delta(t) \geqslant \frac{1}{N}\int_{|v| \leqslant \delta(t)} dF_t(v) \geqslant 1 - (\overline{T}/\overline{T}_c)^{3/5}, \tag{7.9}$$

where $C = 5\pi/(bN)$ and

$$\delta(t) = (\|f_t - \Omega_{1,b}\|_{L^1_2(\setminus 0)})^{1/4} \to 0 \quad (t \to \infty)\,.$$

Remark. In the case $\mu_0 = 0$, i.e. there is no velocity concentration at the initial state, the theorem insures that the single point concentration still occurs as $t \to \infty$ even f_0 is smooth. On the other hand, for the case $f_0 = 0$, i.e. F_0 is totaly singular (with respect to the Lebesgue measure), there is so far no result on the long time behavior of the corresponding solution F_t .

Proof of Theorem 2. Part (I): We need to construct approximate L^1-solutions f^n of Eq. (BBE) with initial data f_0^n which satisfy the condition (6.2)–(6.3) in Theorem 1 for $K = n$. To do this we consider $d\nu_0(v) = \frac{1}{2}f_0(v)dv + d\mu_0(v)$. By assumption on f_0 we have $\int_{\mathbf{R}^3}|v|^2 d\nu_0(v) > 0$ so that the Mehler transform can be used to construct isotropic functions $0 \leqslant h_0^n \in L^1_2(\mathbf{R}^3)$ satisfying (see Lemma 6 in ref. 22)

$$\int_{\mathbf{R}^3}(1, v, |v|^2)h_0^n(v)dv = \int_{\mathbf{R}^3}(1, v, |v|^2)d\nu_0(v)\,, \quad n = 1, 2, 3, \ldots$$

$$\lim_{n\to\infty}\int_{\mathbf{R}^3}\varphi(v)h^n(v)dv = \int_{\mathbf{R}^3}\varphi(v)d\nu_0(v) \quad \forall \varphi \in C \cap L^\infty_{-2}(\mathbf{R}^3)\,.$$

Here $C\cap L^\infty_{-2}(\mathbf{R}^3)$ is given in (4.7). Let $f_0^n(v)=\frac{1}{2}f_0(v)+h_0^n(v)$. Then

$$\int_{\mathbf{R}^3}(1,v,|v|^2)f_0^n(v)dv=(N,0,E)\,,\quad n=1,2,3,\ldots \tag{7.10}$$

$$\lim_{n\to\infty}\int_{\mathbf{R}^3}\varphi(v)f_0^n(v)dv=\int_{\mathbf{R}^3}\varphi(v)dF_0(v)\quad \forall\varphi\in C\cap L^\infty_{-2}(\mathbf{R}^3)\,.$$

Moreover by monotonicity (4.3), $f_0^n\geqslant\frac{1}{2}f_0$, and $\int_{\mathbf{R}^3}f_0(v)dv>0$ we have

$$\inf_{n\geqslant 1}S(f_0^n)\geqslant S(\frac{1}{2}f_0)>0\,. \tag{7.11}$$

For any n, let $B_n(v-v_*,\omega)$ be the cutoff kernel given in (6.1) with $K=n$, and let f^n be the unique conservative isotropic solution of Eq. (BBE) with the kernels B_n and $f^n|_{t=0}=f_0^n$. By the weak stability and the existence results proved in ref. 22, there exist a subsequence $\{f^{n_j}\}_{j=1}^\infty$ of $\{f^n\}$ and a conservative isotropic distributional solution F_t of Eq. (BBE) with $F_t|_{t=0}=F_0$, such that (7.4) holds. Also by (7.10)–(7.11) and Theorem 1, $\{f^n\}$ has all properties listed in Theorem 1 for $K=n\in\mathcal{K}=\{1,2,3,\ldots\}$.

Part (II). Let $dF_t(v)=f_t(v)dv+d\mu_t(v)$ be the Lebesgue decomposition as stated in the theorem. Since F_t conserves the mass and energy, it follows from Proposition 1 that for any temperature and for all $t\geqslant 0$

$$\int_{\mathbf{R}^3}|v|^2f_t(v)dv\leqslant\int_{\mathbf{R}^3}|v|^2dF_t(v)=E=\int_{\mathbf{R}^3}|v|^2\Omega_{a,b}(v)dv\,. \tag{7.12}$$

Moreover $\overline{T}<\overline{T}_c$ implies $a=1$; $\overline{T}\geqslant\overline{T}_c$ implies that for all $t\geqslant 0$

$$\int_{\mathbf{R}^3}f_t(v)dv\leqslant\int_{\mathbf{R}^3}dF_t(v)=N=\int_{\mathbf{R}^3}\Omega_{a,b}(v)dv\,. \tag{7.13}$$

Therefore the conditions in Part (2) of Lemma 4 are all satisfied for f_t and $\Omega_{a,b}$. Furthermore, the equality in (7.12) implies that

$$\int_{\mathbf{R}^3}|v|^2d\mu_t(v)=\int_{\mathbf{R}^3}|v|^2(\Omega_{a,b}-f_t)dv\leqslant\|f_t-\Omega_{a,b}\|_{L^1_2(\backslash 0)}\,, \tag{7.14}$$

while the equalities in (7.12) and (7.13) imply

$$\int_{\mathbf{R}^3}(1+|v|^2)d\mu_t(v)=\int_{\mathbf{R}^3}(1+|v|^2)(\Omega_{a,b}-f_t)dv\leqslant \|f_t-\Omega_{a,b}\|_{L^1_2}\,. \tag{7.15}$$

Thus by Lemma 4 ((4.14)–(4.16)) and (7.14)–(7.15) we conclude that the estimates (7.5), (7.6) (for any temperature) and (7.8) (for $\overline{T}\geqslant\overline{T}_c$) hold true.

Next we prove (7.7). By weak convergence (7.4) and Lemma 3 we have

$$\limsup_{j\to\infty} S(f^{n_j}(t))\leqslant S(f_t)\leqslant S(\Omega_{a,b})\quad \forall t\geqslant 0\,.$$

By Theorem 1, this implies

$$0\leqslant S(\Omega_{a,b})-S(f_t)\leqslant \sup_{n\geqslant 1}[S(\Omega_{a,b})-S(f^n(t))]\to 0\quad (t\to\infty)\,.$$

Finally we prove (7.9). Suppose $\overline{T}<\overline{T}_c$. Then $a=1$. Let $\delta(t)$ be given in the theorem. Note that if $\delta(t)=0$ for some $t\geqslant 0$, then $f_t=\Omega_{1,b}$ a.e. on $\mathbf{R}^3$, so that $dF_t(v)=\Omega_{1,b}(v)dv+d\mu_t(v)$, and by (7.14) the measure μ_t concentrates at $v=0$. Thus by conservation of mass we deduce that

$$N=\int_{\mathbf{R}^3}dF_t(v)=\int_{\mathbf{R}^3}\Omega_{1,b}(v)dv+\mu_t(\{0\})$$

which implies $\mu_t(\{0\})=N_0=N(1-(\overline{T}/\overline{T}_c)^{3/5})$ and hence

$$\frac{1}{N}F_t(\{0\})=\frac{1}{N}\mu_t(\{0\})=1-(\overline{T}/\overline{T}_c)^{3/5}\,.$$

This means that (7.9) holds for $\delta(t)=0$. Therefore to prove (7.9) we can assume that $\delta(t)>0$ for any given t. We compute

$$\int_{|v|\leqslant\delta(t)}dF_t(v)\;=\;N_0+I(t)\,,$$

$$I(t)\;=\;\int_{|v|\leqslant\delta(t)}\Omega_{1,b}dv+\int_{|v|>\delta(t)}(\Omega_{1,b}-f_t)dv-\int_{|v|>\delta(t)}d\mu_t(v)\,,$$

and, by (7.14),

$$\int_{|v|>\delta(t)} d\mu_t(v) \leqslant \frac{1}{\delta^2(t)} \int_{\mathbf{R}^3} |v|^2 d\mu_t(v) \leqslant \frac{1}{\delta^2(t)} \| f_t - \Omega_{1,b} \|_{L^1_2(\setminus 0)} = \delta^2(t)\,.$$

The rest of proof is the same as that for Part (II.3) of Theorem 1 with the same constant $C = 5\pi/(bN)$. ∎

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